Cannon-Thurston Maps for Surface Groups

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Abstract

We prove the existence of Cannon-Thurston maps for simply and doubly degenerate surface Kleinian groups. As a consequence we prove that, for arbitrary finitely generated Kleinian groups, connected limit sets are locally connected.

MSC classification: 57M50

Contents

1   Introduction

In Section 6 of [CT85] [CT07], Cannon and Thurston raise the following question:

Question 1.1. *Suppose that a surface group $\pi_1(S)$ acts freely and properly discontinuously on $\mathbb{H}^3$ by isometries such that the quotient manifold has no accidental parabolics. Does the inclusion $\tilde{i} : \widetilde{S} \to \mathbb{H}^3$ extend continuously to the boundary?*

The authors of [CT85] point out that for a simply degenerate group, this is equivalent, via the Caratheodory extension Theorem, to asking if the limit set is locally connected.

Minsky [Min94], Alperin-Dicks-Porti [ADP99], Cannon-Dicks [CD02] [CD06], Klarreich [Kla99], McMullen [McM01], Bowditch [Bow02] [Bow07] and the author [Mit98b], [Mit98a], [Mj05b], [Mj06a] have obtained partial positive answers. An approach to a negative answer had been indicated by Abikoff in [Abi76]. In this paper, we give a positive answer to the above question.

Theorems 7.1 and 8.6: *Let ρ be a representation of a surface group H (corresponding to the surface S) into $PSl_2(C)$ without accidental parabolics. Let M denote the (convex core of) $\mathbb{H}^3/\rho(H)$. Further suppose that $i : S \to M$, taking parabolic to parabolics, induces a homotopy equivalence. Then the inclusion $\tilde{i} : \widetilde{S} \to \widetilde{M}$ extends continuously to a map of the compactifications $\hat{i} : \widehat{S} \to \widehat{M}$. Hence the limit set of $\widetilde{S}$ is locally connected.*

The continuous boundary extensions above are called Cannon-Thurston maps. The existence of such maps was proven
1) by Cannon and Thurston [CT85] [CT07] for fibers of closed hyperbolic 3 manifolds fibering over the circle and for simply degenerate groups with asymptotically periodic ends.
2) by Alperin-Dicks-Porti [ADP99] for the figure eight knot complement.
3) by Minsky [Min94] for closed surface groups of bounded geometry (see also [Mit98b], [Mj06a]).

4) by McMullen [McM01] for punctured torus groups.
5) by Bowditch [Bow02] [Bow07] for punctured surface groups of bounded geometry (see also [Mj09]).

Combining Theorems 7.1 and 8.6 with a theorem of Anderson and Maskit [AM96], we have the following.

Theorem 8.8: *Let* Γ *be a finitely generated Kleinian group with connected limit set* Λ. *Then* Λ *is locally connected.*

The techniques of this paper can be strengthened further to show that Cannon-Thurston maps exist in general for finitely generated Kleinian groups, thus answering a conjecture of McMullen [McM01]. This is sketched in [Mj05a] and [Mj06b], and we postpone a thorough discussion to a future work. In [Mj07], we prove that the point pre-images of the Cannon-Thurston map for closed surface groups without accidental parabolics are precisely the end-points of leaves of the ending lamination.

The basic strategy in proving the existence of Cannon-Thurston maps is similar to the one we followed in [Mit98b] and [Mit98a]. Given a hyperbolic geodesic segment λ in $\widetilde{S}$ lying outside a large ball about a fixed reference point, we show that the geodesic in $\mathbf{H}^3$ joining its endpoints also lies outside a large ball. An essential tool is the construction of a quasiconvex 'hyperbolic ladder' in an auxiliary hyperbolic metric space.

We recall the notions of relative hyperbolicity and electric geometry (cf. [Far98]) in Sections 2.1, 2.2 and derive some consequences that will be useful in this paper in Sections 2.3-2.5. In Section 3, we collect together features of the model manifold constructed by Minsky in [Min02] and proven to be a bi-Lipschitz model for simply and doubly degenerate manifolds by Brock-Canary-Minsky in [BCM04]. In Section 4, we select out a sequence of split surfaces from the split surfaces occurring in the model manifold and proceed to 'fill' the intermediate spaces between successive split surfaces by special blocks homeomorphic to $S \times I$. This gives us a 'split geometry' model for simply and doubly degenerate manifolds. We make crucial use of electric geometry and relative hyperbolicity at this stage. In Section 5, we construct a quasiconvex 'hyperbolic ladder' in the hyperbolic electric space constructed in Section 4 and use it to construct a quasigeodesic in the electric metric joining the endpoints of λ. In Section 6, we recover information about the hyperbolic geodesic joining the endpoints of λ from the electric geodesic constructed in Section 5. In Section 7 we put all the ingredients together to prove the existence of Cannon-Thurston maps for closed surface Kleinian groups (Theorem 7.1). In Section 8 we describe the modifications necessary for punctured surfaces.

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are due to the referee for not unwelcome pressure to re-organize a couple of unwieldy manuscripts ([Mj05a] and [Mj06b]) into a relatively more streamlined version. Partial support for this project was provided by a DST research grant.

Dedication: This paper is fondly dedicated to Gadai and Sri Sarada for their support and indulgence.

1.1   Hyperbolic Metric Spaces and Cannon-Thurston Maps

We start off with some preliminaries about hyperbolic metric spaces in the sense of Gromov [Gro85]. For details, see [CDA90], [GdlH90]. Let (X, d) be a hyperbolic metric space. The **Gromov boundary** of X, denoted by ∂X, is the collection of equivalence classes of geodesic rays $r : [0, \infty) \to X$ with $r(0) = x_0$ for some fixed $x_0 \in X$, where rays r_1 and r_2 are equivalent if $sup\{d(r_1(t), r_2(t))\} < \infty$. Let $\widehat{X} = X \cup \partial X$ denote the natural compactification of X topologized the usual way(cf.[GdlH90] pg. 124).

Definitions: A subset Z of X is said to be k**-quasiconvex** if any geodesic joining points of Z lies in a k-neighborhood of Z. A subset Z is **quasiconvex** if it is k-quasiconvex for some k. (For simply connected real hyperbolic manifolds this is equivalent to saying that the convex hull of the set Z lies in a bounded neighborhood of Z. We shall have occasion to use this alternate characterization.) A map f from one metric space (Y, d_Y) into another metric space (Z, d_Z) is said to be a (K, ϵ)**-quasi-isometric embedding** if

$$\tfrac{1}{K}(d_Y(y_1, y_2)) - \epsilon \leq d_Z(f(y_1), f(y_2)) \leq K d_Y(y_1, y_2) + \epsilon$$

If f is a quasi-isometric embedding, and every point of Z lies at a uniformly bounded distance from some $f(y)$ then f is said to be a **quasi-isometry**. A (K, ϵ)-quasi-isometric embedding that is a quasi-isometry will be called a (K, ϵ)-quasi-isometry.

A (K, ϵ)**-quasigeodesic** is a (K, ϵ)-quasi-isometric embedding of a closed interval in $\mathbb{R}$. A (K, K)-quasigeodesic will also be called a K-quasigeodesic.

Let (X, d_X) be a proper hyperbolic metric space and Y be a subspace that is hyperbolic with the inherited path metric d_Y. By adjoining the Gromov boundaries ∂X and ∂Y to X and Y, one obtains their compactifications $\widehat{X}$ and $\widehat{Y}$ respectively.

Let $i : Y \to X$ denote inclusion.

Definition: Let X and Y be hyperbolic metric spaces and $i : Y \to X$ be an embedding. A **Cannon-Thurston map** $\hat{i}$ from $\widehat{Y}$ to $\widehat{X}$ is a continuous extension of i.

The following lemma (Lemma 2.1 of [Mit98a]) says that a Cannon-Thurston map exists if for all $M > 0$ and $y \in Y$, there exists $N > 0$ such that if λ lies outside an N ball around y in Y then any geodesic in X joining the end-points of λ lies outside the M ball around $i(y)$ in X. For convenience of use later on, we state this somewhat differently.

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Lemma 1.2. *A Cannon-Thurston map from $\widehat{Y}$ to $\widehat{X}$ exists if the following condition is satisfied:*

Given $y_0 \in Y$, there exists a non-negative function $M(N)$, such that $M(N) \to \infty$ as $N \to \infty$ and for all geodesic segments λ lying outside an N-ball around $y_0 \in Y$ any geodesic segment in Γ_G joining the end-points of $i(\lambda)$ lies outside the $M(N)$-ball around $i(y_0) \in X$.

The above result can be interpreted as saying that a Cannon-Thurston map exists if the space of geodesic segments in Y embeds properly in the space of geodesic segments in X.

2 Relative Hyperbolicity

In this section, we shall recall first certain notions of relative hyperbolicity due to Farb [Far98], Klarreich [Kla99] and the author [Mj05b]. Using these, we shall derive certain Lemmas that will be useful in studying the geometry of the universal covers of building blocks (see below).

2.1 Electric Geometry

We collect together certain facts about the electric metric that Farb proves in [Far98]. $N_R(Z)$ will denote the R-neighborhood about the subset Z in the hyperbolic metric. $N_R^e(Z)$ will denote the R-neighborhood about the subset Z in the electric metric.

Given a metric space X and a collection $\mathcal{H}$ of subsets we define the **electric space** $\mathcal{E}(X, \mathcal{H})$ to be the space obtained by attaching, for each $H \in \mathcal{H}$ metric products $H \times [0, 1]$ to X such that $(h, 0)$ is identified with $h \in X$ and putting the zero metric on $H \times \{1\}$.

We shall mostly consider a hyperbolic metric space X and a collection $\mathcal{H}$ of *(uniformly) C-quasiconvex uniformly D-separated subsets*, i.e. there exists $D > 0$ such that for $H_1, H_2 \in \mathcal{H}$, $d_X(H_1, H_2) \geq D$. In this situation X is (weakly) hyperbolic relative to the collection $\mathcal{H}$, i.e. $\mathcal{E}(X, \mathcal{H})$ is a hyperbolic metric space. The result in this form is due to Bowditch [Bow97] (See also Klarreich [Kla99]). We give the general version of Farb's theorem below in Lemma 2.1 and refer to [Far98] and Klarreich [Kla99] for proofs.

Definitions: Given a collection $\mathcal{H}$ of C-quasiconvex, D-separated sets and a number ϵ we shall say that a geodesic (resp. quasigeodesic) γ is a geodesic (resp. quasigeodesic) **without backtracking** with respect to ϵ neighborhoods if γ does not return to $N_\epsilon(H)$ after leaving it, for any $H \in \mathcal{H}$. A geodesic (resp. quasigeodesic) γ is a geodesic (resp. quasigeodesic) **without backtracking** if it is a geodesic (resp. quasigeodesic) without backtracking with respect to ϵ neighborhoods for some $\epsilon \geq 0$.

Note: For strictly convex sets, $\epsilon = 0$ suffices, whereas for convex sets any $\epsilon > 0$ is enough.

Lemma 2.1. *(See Lemma 4.5 and Proposition 4.6 of [Far98] and Theorem 5.3 of Klarreich [Kla99]) Given δ, C, D there exists Δ such that if X is a δ-hyperbolic metric space with a collection $\mathcal{H}$ of C-quasiconvex D-separated sets. then,*

1. *Electric quasi-geodesics electrically track hyperbolic geodesics: Given $P > 0$, there exists $K > 0$ with the following property: Let β be any electric P-quasigeodesic from x to y, and let γ be the hyperbolic geodesic from x to y. Then $\beta \subset N_K^e(\gamma)$.*

2. *γ lies in a hyperbolic K-neighborhood of $N_0(\beta)$, where $N_0(\beta)$ denotes the zero neighborhood of β in the electric metric.*

3. *Hyperbolicity: X is Δ-hyperbolic.*

Let X be a δ-hyperbolic metric space, and $\mathcal{H}$ a family of C-quasiconvex, D-separated, collection of subsets. Then by Lemma 2.1, $X_{el} = \mathcal{E}(X, \mathcal{H})$ obtained by electrocuting the subsets in $\mathcal{H}$ is a $\Delta = \Delta(\delta, C, D)$ -hyperbolic metric space. Now, let $\alpha = [a, b]$ be a hyperbolic geodesic in X and β be an electric P-quasigeodesic without backtracking joining a, b. Replace each maximal subsegment, (with end-points p, q, say) starting from the left of β lying within some $H \in \mathcal{H}$ by a hyperbolic geodesic $[p, q]$. The resulting **connected** path β_q is called an *electro-ambient representative* in X.

Note that β_q *need not be a hyperbolic quasigeodesic.* However, the proof of Proposition 4.3 of Klarreich [Kla99] gives the following:

Lemma 2.2. *(See Proposition 4.3 of [Kla99], also see Lemma 3.10 of [Mj05b]) Given δ, C, D, P there exists C_3 such that the following holds:*
Let (X, d) be a δ-hyperbolic metric space and $\mathcal{H}$ a family of C-quasiconvex, D-separated collection of quasiconvex subsets. Let (X, d_e) denote the electric space obtained by electrocuting elements of $\mathcal{H}$. Then, if α, β_q denote respectively a hyperbolic geodesic and an electro-ambient P-quasigeodesic with the same endpoints, then α lies in a (hyperbolic) C_3 neighborhood of β_q.

2.2   Coboundedness and Consequences

In this Subsection, we collect together a few more results that strengthen Lemma 2.1.

Definition: A collection $\mathcal{H}$ of uniformly C-quasiconvex sets in a δ-hyperbolic metric space X is said to be **mutually D-cobounded** if for all $H_i, H_j \in \mathcal{H}$, $\pi_i(H_j)$ has diameter less than D, where π_i denotes a nearest point projection of X onto H_i. A collection is **mutually cobounded** if it is mutually D-cobounded for some D.

Lemma 2.3. *Suppose X is a δ-hyperbolic metric space with a collection $\mathcal{H}$ of C-quasiconvex K-separated D-mutually cobounded subsets. There exists $\epsilon_0 = \epsilon_0(C, K, D, \delta)$ such that the following holds:*

Let β be an electric P-quasigeodesic without backtracking and γ a hyperbolic geodesic, both joining x, y. Then, given $\epsilon \geq \epsilon_0$ there exists $D = D(P, \epsilon)$ such that

1. *Similar Intersection Patterns 1: if precisely one of $\{\beta, \gamma\}$ meets an ϵ-neighborhood $N_\epsilon(H_1)$ of an electrocuted quasiconvex set $H_1 \in \mathcal{H}$, then the length (measured in the intrinsic path-metric on $N_\epsilon(H_1)$) from the entry point to the exit point is at most D.*

2. *Similar Intersection Patterns 2: if both $\{\beta, \gamma\}$ meet some $N_\epsilon(H_1)$ then the length (measured in the intrinsic path-metric on $N_\epsilon(H_1)$) from the entry point of β to that of γ is at most D; similarly for exit points.*

Summarizing, we have:

• If X is a hyperbolic metric space and $\mathcal{H}$ a collection of uniformly quasiconvex mutually cobounded separated subsets, then X is hyperbolic relative to the collection $\mathcal{H}$ and satisfies *Bounded Penetration*, i.e. hyperbolic geodesics and electric quasigeodesics have similar intersection patterns in the sense of Lemma 2.3.

The relevance of co-boundedness comes from the following Lemmas which are essentially due to Farb [Far98].

Lemma 2.4. *Let M^h be a hyperbolic manifold, with Margulis tubes $T_i \in \mathcal{T}$ and horoballs $H_j \in \mathcal{H}$. Then the lifts $\widetilde{T_i}$ and $\widetilde{H_j}$ are mutually co-bounded.*

Lemma 2.5. *Let S^h be a hyperbolic surface, with a finite collection of disjoint simple closed geodesics $\sigma_i \in \mathcal{S}$ and horoballs $H_j \in \mathcal{H}$. Then the entire collection of lifts $\widetilde{\sigma_i}$ and $\widetilde{H_j}$ are mutually co-bounded.*

The proof given in [Far98] is for a collection of separated horospheres, but the same proof works for
a) neighborhoods of geodesics and horospheres (Lemma 2.4)
b) Lifts of a simple closed geodesic on a surface (Lemma 2.5)

A closely related theorem was proved by McMullen (Theorem 8.1 of [McM01]).
As usual, $N_R(Z)$ will denote the R-neighborhood of the set Z.
Let $\mathcal{H}$ be a locally finite collection of horoballs in a convex subset X of $\mathbb{H}^n$ (where the intersection of a horoball, which meets ∂X in a point, with X is called a horoball in X).

Definition: The ϵ-neighborhood of a bi-infinite geodesic in $\mathbb{H}^n$ will be called a **thickened geodesic**.

Theorem 2.6. *[McM01] Let $\gamma : I \to X \backslash \bigcup \mathcal{H}$ be an ambient (K, ϵ)-quasigeodesic (for X a convex subset of $\mathbb{H}^n$) and let $\mathcal{H}$ denote a uniformly separated collection of horoballs and thickened geodesics. Let η be the hyperbolic geodesic with the same endpoints as γ. Let $\mathcal{H}(\eta)$ be the union of all the horoballs and thickened geodesics in $\mathcal{H}$ meeting η. Then $\eta \cup \mathcal{H}(\eta)$ is (uniformly) quasiconvex and $\gamma(I) \subset B_R(\eta \cup \mathcal{H}(\eta))$, where R depends only on K, ϵ.*

2.3 Electric Geometry for Surfaces

We now specialize to surfaces. We start with a surface S (assumed hyperbolic for the time being) of (K, ϵ) bounded geometry, i.e. S has diameter bounded by K and injectivity radius bounded below by ϵ. Let σ be a simple closed geodesic on S. $S - \sigma$ has one or two components according as σ does not or does separate S. Call these **amalgamation component(s)** of S We shall denote amalgamation components as S_A. Let $S_G = \mathcal{E}(S, S_A)$ be obtained by electrocuting S_A's. Then

• the length of any path that lies in the interior of an amalgamation component is zero

• the length of any path that crosses σ once has length one

• the length of any other path is the sum of lengths of pieces of the above two kinds.

This allows us to define distances by taking the infimum of lengths of paths joining pairs of points and gives us a path pseudometric, which we call the **electric metric** on S_G. The electric metric also allows us to define geodesics. Let us call S_G equipped with the above pseudometric (S_{Gel}, d_{Gel}).

Important Note: We may and shall regard $\pi_1(S)$ as a graph of groups with vertex group(s) the subgroup(s) corresponding to amalgamation component(s) and edge group Z, the fundamental group of σ. Then $\widetilde{S}$ equipped with the lift of the above pseudometric is quasi-isometric to the tree corresponding to the splitting on which $\pi_1(S)$ acts.

Paths in S_{Gel} and $\widetilde{S_{Gel}}$ will be called electric paths (following Farb [Far98]). Geodesics and quasigeodesics in the electric metric will be called electric geodesics and electric quasigeodesics respectively.

Definition:

• γ is said to be an electric K, ϵ-quasigeodesic in $\widetilde{S_{Gel}}$ **without backtracking** if γ is an electric K-quasigeodesic in $\widetilde{S_{Gel}}$ and γ does not return to any any lift $\widetilde{S_A} \subset \widetilde{S_{Gel}}$ (of an amalgamation component $S_A \subset S$) after leaving it.

A special kind of *geodesic without backtracking* will be necessary for universal covers $\widetilde{S_{Gel}}$ of surfaces with some electric metric.

Let λ_e be an electric geodesic in some $(\widetilde{S_{Gel}}, d_{Gel})$. Each segment of λ_e between two lifts $\widetilde{\sigma_1}$ and $\widetilde{\sigma_2}$ of σ (lying inside a single lift of an amalgamation component) is required to be perpendicular to the bounding geodesics. We

shall refer to these segments of λ_e as **amalgamation segments** because they lie inside lifts of the amalgamation components.

Let a, b be the points at which λ_e enters and leaves a lift $\widetilde{\sigma}$ of σ. Join a, b by the geodesic subsegment of $\widetilde{\sigma}$ containing them. Such pieces shall be referred to as **interpolating segments**.

The union of the *amalgamation segments* along with the *interpolating segments* gives rise to a preferred representative of a quasigeodesic without backtracking joining the end-points of λ_{Gel}. Such a representative of the class of λ_{Gel} shall be called the **canonical representative** of λ_{Gel}. Further, the underlying set of the canonical representative in the *hyperbolic metric* shall be called the **electro-ambient representative** λ_q of λ_e. Since λ_q turns out to be a hyperbolic quasigeodesic (Lemma 2.7 below), we shall also call it an **electro-ambient quasigeodesic**. See Figure below:
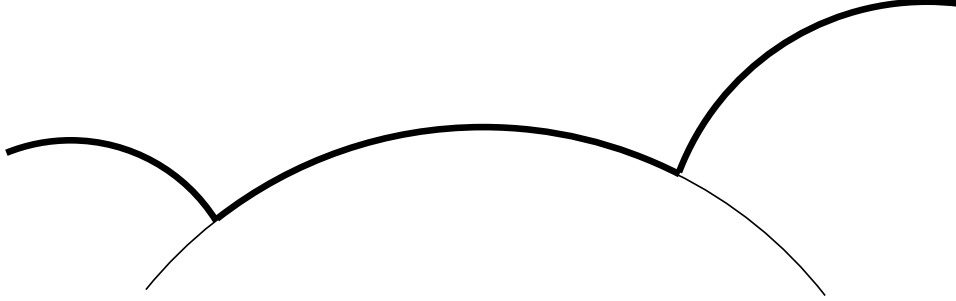

Figure 1:*Electro-ambient quasigeodesic*

Now, let λ_h denote the hyperbolic geodesic joining the end-points of λ_e. By Lemma 2.3, λ_h and λ_e, and hence λ_h and λ_q have similar intersection patterns with $N_\epsilon(\eta)$ for electrocuted geodesics η. Also, λ_h and λ_q track each other off $N_\epsilon(\eta)$. Further, each *interpolating segment* of λ_q being a *hyperbolic* geodesic, it follows (from the 'K-fellow-traveller' property of hyperbolic geodesics starting and ending near each other) that each interpolating segment of λ_q lies within a $(K + 2\epsilon)$ neighborhood of λ_h. Again, since each segment of λ_q that does not meet an electrocuted geodesic that λ_h meets is of uniformly bounded (by C say) length, we have finally that λ_q lies within a $(K + C + 2\epsilon)$ neighborhood of λ_h. Finally, since λ_q is an electro-ambient representative, it does not backtrack. Hence we have the following:

Lemma 2.7. *(See Lemma 3.7 of [Mj05b]) There exists (K, ϵ) such that each electro-ambient representative λ_{Gel} of an electric geodesic in $\widetilde{S_{Gel}}$ is a (K, ϵ) hyperbolic quasigeodesic.*

Proof: Let S_{el} denote the surface S with the geodesic σ electrocuted. Note that the electro-ambient quasigeodesics in $\widetilde{S_{Gel}}$ coincide with those in $\widetilde{S_{el}}$. Hence it suffices to show that electro-ambient quasigeodesics in $\widetilde{S_{el}}$ are uniform hyperbolic quasigeodesics.

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Let λ_h denote the hyperbolic geodesic joining the end-points of λ_e. By Lemmas 2.3 and 2.5, λ_h and λ_e, and hence λ_h and λ_q have similar intersection patterns with $N_\epsilon(\widetilde{\sigma}))$ for some small $\epsilon > 0$. Also, λ_h and λ_q track each other off $N_\epsilon(\widetilde{\sigma})$. Further, each *interpolating segment* of λ_q being a *hyperbolic* geodesic, it follows (from the 'K-fellow-traveller' property of hyperbolic geodesics starting and ending near each other) that each interpolating segment of λ_q lies within a $(K + 2\epsilon)$ neighborhood of λ_h for some fixed $K > 0$. Again, since each segment of λ_q that does not meet an electrocuted geodesic that λ_h meets is of uniformly bounded (by C say) length, we have finally that λ_q lies within a $(K + C + 2\epsilon)$ neighborhood of λ_h. Finally, since λ_q is an electro-ambient representative, it does not backtrack. Hence the Lemma. $\square$

2.4 Electric isometries

Let ϕ be any diffeomorphism of S that fixes σ pointwise and (in case $(S - \sigma)$ has two components) preserves each amalgamation component as a set, i.e. ϕ sends each amalgamation component to itself. Such a ϕ will be called a **component preserving diffeomorphism**. Then in the electrocuted surface S_{Gel}, any electric geodesic has length equal to the number of times it crosses σ. It follows that ϕ is an isometry of S_{Gel}.

Lemma 2.8. *Let ϕ denote a component preserving diffeomorphism of S_G. Then ϕ induces an isometry of (S_{Gel}, d_{Gel}).*

Everything in the above can be lifted to the universal cover $\widetilde{S_{Gel}}$. We let $\widetilde{\phi}$ denote the lift of ϕ to $\widetilde{S_{Gel}}$. This gives

Lemma 2.9. *Let $\widetilde{\phi}$ denote a lift of a component preserving diffeomorphism ϕ to $(\widetilde{S_{Gel}}, d_{Gel})$. Then $\widetilde{\phi}$ induces an isometry of $(\widetilde{S_{Gel}}, d_{Gel})$.*

2.5 Nearest-point Projections

We need the following basic lemmas from [Mit98b].

The next Lemma says nearest point projections in a δ-hyperbolic metric space do not increase distances much.

Lemma 2.10. *(Lemma 3.1 of [Mit98b]) Let (Y, d) be a δ-hyperbolic metric space and let $\mu \subset Y$ be a C-quasiconvex subset, e.g. a geodesic segment. Let $\pi : Y \to \mu$ map $y \in Y$ to a point on μ nearest to y. Then $d(\pi(x), \pi(y)) \leq C_3 d(x, y)$ for all $x, y \in Y$ where C_3 depends only on δ, C.*

The next lemma says that quasi-isometries and nearest-point projections on hyperbolic metric spaces 'almost commute'.

Lemma 2.11. *(Lemma 3.5 of [Mit98b]) Suppose (Y_1, d_1) and (Y_2, d_2) are δ-hyperbolic. Let μ_1 be some geodesic segment in Y_1 joining a, b and let p be any vertex of Y_1. Also let q be a vertex on μ_1 such that $d_1(p, q) \leq d_2(p, x)$ for*

$x \in \mu_1$. Let ϕ be a (K, ϵ) - *quasiisometric embedding from* Y_1 *to* Y_2. *Let* μ_2 *be a geodesic segment in* Y_2 *joining* $\phi(a)$ *to* $\phi(b)$. *Let* r *be a point on* μ_2 *such that* $d_2(\phi(p), r) \leq d_2(\phi(p), x)$ *for* $x \in \mu_2$. *Then* $d_2(r, \phi(q)) \leq C_4$ *for some constant* C_4 *depending only on* K, ϵ *and* δ.

For our purposes we shall need the above Lemma for quasi-isometries from $\widetilde{S_a}$ to $\widetilde{S_b}$ for two different hyperbolic structures on the same surface. We shall also need it for electrocuted surfaces.

Yet another property that we shall require for nearest point projections is that nearest point projections in the electric metric and in the hyperbolic metric almost agree. Equip $\widetilde{S}$ with the path metric d as usual. Recall that d_{Gel} denotes the electric metric on $Y = \widetilde{S_G}$ obtained by electrocuting the lifts of complementary components. Now, let $\mu = [a, b]$ be an electric geodesic on (Y, d) and let μ_q denote the electro-ambient quasigeodesic joining a, b (See Lemma 2.7). Let π denote the nearest point projection in (Y, d). Tentatively, let π_e denote the nearest point projection in (Y, d_{Gel}). Note that π_e is not well-defined. It is defined up to a bounded amount of discrepancy in the electric metric d_e. But we would like to make π_e well-defined up to a bounded amount of discrepancy in the metric d.

Definition: Let $y \in Y$ and let μ_q be an electro-ambient representative of an electric geodesic μ_{Gel} in (Y, d_{Gel}). Then $\pi_e(y) = z \in \mu_q$ if the ordered pair $\{d_{Gel}(y, \pi_e(y)), d(y, \pi_e(y))\}$ is minimized at z.

The proof of the following Lemma shows that this gives us a definition of π_e which is ambiguous by a finite amount of discrepancy not only in the electric metric but also in the hyperbolic metric.

Lemma 2.12. *There exists* $C > 0$ *such that the following holds. Let* μ *be a hyperbolic geodesic joining* a, b. *Let* μ_{Gel} *be an electric geodesic joining* a, b. *Also let* μ_q *be the electro-ambient representative of* μ_{Gel}. *Let* π_h *denote the nearest point projection of* Y *onto* μ. $d(\pi_h(y), \pi_e(y))$ *is uniformly bounded.*

Proof: $[u, v]$ and $[u, v]_q$ will denote respectively the hyperbolic geodesic and the electro-ambient quasigeodesic joining u, v. Since $[u, v]_q$ is a quasigeodesic by Lemma 2.7, it suffices to show that for any y, its hyperbolic and electric projections $\pi_h(y), \pi_e(y)$ almost agree.

First note that any hyperbolic geodesic η in $\widetilde{S}$ is also an electric geodesic. This follows from the fact that $(\widetilde{S_G}, d_{Gel})$ maps to a tree T (arising from the splitting along σ) with the pullback of every vertex a set of diameter zero in the pseudometric d_{Gel}. Now if a path in $\widetilde{S_G}$ projects to a path in T that is not a geodesic, then it must backtrack. Hence, it must leave an amalgamating component and return to it. Such a path can clearly not be a hyperbolic geodesic in $\widetilde{S_G}$ (since each amalgamating component is convex).

Next, it follows that hyperbolic projections automatically minimize electric distances. Else as in the preceding paragraph, $[y, \pi_h(y)]$ would have to cut a lift of $\widetilde{\sigma} = \widetilde{\sigma_1}$ that separates $[u, v]_q$. Further, $[y, \pi_h(y)]$ cannot return to $\widetilde{\sigma_1}$ after leaving it. Let z be the first point at which $[y, \pi_h(y)]$ meets $\widetilde{\sigma_1}$. Also let w be the

point on $[u, v]_q \cap \widetilde{\sigma_1}$ that is nearest to z. Since amalgamation segments of $[u, v]_q$ meeting $\widetilde{\sigma_1}$ are perpendicular to the latter, it follows that $d(w, z) < d(w, \pi_h(y))$ and therefore $d(y, z) < d(y, \pi_h(y))$ contradicting the definition of $\pi_h(y)$. Hence hyperbolic projections automatically minimize electric distances.

Further, it follows by repeating the argument in the first paragraph that $[y, \pi_h(y)]$ and $[y, \pi_e(y)]$ pass through the same set of amalgamation components in the same order; in particular they cut across the same set of lifts of $\widetilde{\sigma}$. Let $\widetilde{\sigma_2}$ be the last such lift. Then $\widetilde{\sigma_2}$ forms the boundary of an amalgamation component $\tilde{S_A}$ whose intersection with $[u, v]_q$ is of the form $[a, b] \cup [b, c] \cup [c, d]$, where $[a, b] \subset \widetilde{\sigma_3}$ and $[c, d] \subset \widetilde{\sigma_4}$ are subsegments of two lifts of σ and $[b, c]$ is perpendicular to these two. Then the nearest-point projection of $\widetilde{\sigma_2}$ onto each of $[a, b], [b, c], [c, d]$ has uniformly bounded diameter. Hence the nearest point projection of $\tilde{\sigma}_2$ onto the hyperbolic geodesic $[a, d] \subset \widetilde{S_A}$ has uniformly bounded diameter. The result follows. $\square$

3   The Minsky Model

Fix a hyperbolic structure on a Riemann surface S and construct the metric product $S \times \mathbb{R}$. Fix further a positive real number l_0.

Definition 3.1. *An annulus A will be said to be **vertical** if it is of the form $\sigma \times J$ for σ a geodesic of length less than l_0 on S and $J = [a, b]$ a closed sub-interval of $\mathbb{R}$. J will be called the **vertical interval** for the vertical annulus A. A disjoint collection of annuli is said to be a **vertical system** of annuli if each annulus in the collection is vertical.*

The above definition is based on a definition due to Bowditch [Bow05a],[Bow05b]. See figure below.
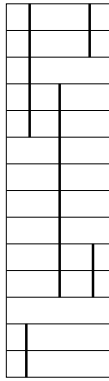

Figure 2: *Vertical Annulus Structure*

A slight modification of the vertical annulus structure will sometimes be useful.

Replacing each geodesic γ on S by a neighborhood $N_\epsilon(\gamma)$ for sufficiently small ϵ, we obtain a **vertical Margulis tube** structure after taking products with vertical intervals. The family of Margulis tubes shall be denoted by $\mathcal{T}$ and the union of their interiors as $Int\mathcal{T}$. The union of $Int\mathcal{T}$ and its horizontal boundaries (corresponding to neighborhoods of geodesics $\gamma \subset S$) shall be denoted as $Int^+\mathcal{T}$.

3.1   Tight Geodesics and Hierarchies

In this subsection we collect together the necessary notions and facts from Minsky [Min02]. Let $\mathcal{C}(S)$ denote the curve complex for a surface S with the usual modifications for surfaces of small complexity. We shall mostly be concerned with only the pants complex in what follows rather than the marking complex as in [Min02]. This simplifies the discussion somewhat as we are interested in tight geodesics in subsurfaces of complexity ≥ 4 (see below).

A marking μ is a collection of base curves on S which form a simplex in $\mathcal{C}(S)$. For an essential subsurface $W \subset S$, the restriction $\mu|_W$ of μ to W consists of those curves which meet W essentially.

Fix a hyperbolic structure on S. If v is a simplex of $\mathcal{C}(S)$, γ_v will denote its geodesic representative on S with the fixed hyperbolic structure.

The **complexity** of a compact surface $S_{g,b}$ of genus g and b boundary components is defined to be $\xi(S_{g,b}) = 3g + b$.

Tight geodesics

A pair of simplices α, β in a $\mathcal{C}(Y)$ are said to *fill* Y if all non-trivial non-peripheral curves in Y intersect at least one of γ_α or γ_β.

Given arbitrary simplices α, β in $\mathcal{C}(S)$, form a regular neighborhood of $\gamma_\alpha \cup \gamma_\beta$, and and fill in all disks and one-holed disks to obtain Y which is filled by α, β.

For a subsurface $X \subseteq Z$ let $\partial_Z(X)$ denote the *relative boundary* of X in Z, i.e. those boundary components of X that are non-peripheral in Z.

Definition 3.2. *Let Y be an essential subsurface in S. If $\xi(Y) > 4$, a sequence of simplices $\{v_i\}_{i\in\mathcal{I}} \subset \mathcal{C}(Y)$ (where $\mathcal{I}$ is a finite or infinite interval in $\mathbb{Z}$) is called* tight *if*
1) For any vertices w_i of v_i and w_j of v_j where $i \neq j$, $d_{\mathcal{C}_1(Y)}(w_i, w_j) = |i - j|$,
2) Whenever $\{i-1, i, i+1\} \subset \mathcal{I}$, v_i represents the relative boundary $\partial_Y F(v_{i-1}, v_{i+1})$.

If $\xi(Y) = 4$ then a tight sequence is the vertex sequence of a geodesic in $\mathcal{C}(Y)$.

A tight geodesic g in $\mathcal{C}(Y)$ consists of a tight sequence $v_0, \cdots, v_n$, and two simplices in $\mathcal{C}(Y)$, $\mathbf{I} = \mathbf{I}(g)$ and $\mathbf{T} = \mathbf{T}(g)$, called its initial *and* terminal *markings such that v_0 (resp. v_n) is a sub-simplex of $\mathbf{I}$ (resp. $\mathbf{T}$). The length of g is n.*

v_i *is called a* simplex *of g. Y is called the* domain *or* support of g *and is denoted as $Y = D(g)$. g is said to be* supported in $D(g)$.

We denote the obvious linear order in g as $v_i < v_j$ whenever $i < j$.

A geodesic supported in Y with $\xi(Y) = 4$ is called a 4-geodesic.

If v_i is a simplex of g define its *successor*

$$\mathrm{succ}(v_i) = \begin{cases} v_{i+1} & v_i \text{ is not the last simplex} \\ \mathbf{T}(g) & v_i \text{ is the last simplex} \end{cases}$$

and similarly define $\mathrm{pred}(v_i)$ to be v_{i-1} or $\mathbf{I}(g)$.

Definition 3.3. Component domains:
Given a surface W with $\xi(W) \geq 4$ and a simplex v in $\mathcal{C}(W)$ we say that Y is a component domain of (W, v) if Y is a component of $W \setminus \mathbf{collar}(v)$.

If g is a tight geodesic with domain $D(g)$, we call $Y \subset S$ a component domain of g if for some simplex v_j of g, Y is a component domain of $(D(g), v_j)$. We note that g and Y determine v_j uniquely. In such a case, let

$$\mathbf{I}(Y, g) = \mathrm{pred}(v_j)|_Y,$$

$$\mathbf{T}(Y, g) = \mathrm{succ}(v_j)|_Y.$$

If Y is a component domain of g and $\mathbf{T}(Y, g) \neq \emptyset$ then we say that Y is *directly forward subordinate* to g, or $Y \searrow^d g$. Similarly if $\mathbf{I}(Y, g) \neq \emptyset$ we say that Y is *directly backward subordinate* to g, or $g \swarrow^d Y$.

Definition 3.4. *If k and g are tight geodesics, we say that k is directly forward subordinate to g, or $k \searrow^d g$, provided $D(k) \searrow^d g$ and $\mathbf{T}(k) = \mathbf{T}(D(k), g)$. Similarly we define $g \swarrow^d k$ to mean $g \swarrow^d D(k)$ and $\mathbf{I}(k) = \mathbf{I}(D(k), g)$.*

We denote by *forward-subordinate*, or $\searrow$, the transitive closure of $\searrow^d$, and similarly for $\swarrow$. We let $h \unrhd k$ denote the condition that $h = k$ or $h \searrow k$, and similarly for $k \unlhd h$. We include the notation $Y \searrow f$ where Y is a subsurface to mean $Y \searrow^d f'$ for some f' such that $f' \unrhd f$, and similarly define $b \swarrow Y$.

Definition 3.5. *A **hierarchy of geodesics** is a collection H of tight geodesics in S with the following properties:*

1. *There is a distinguished* main geodesic g_H *with domain $D(g_H) = S$. The initial and terminal markings of g_H are denoted also $\mathbf{I}(H), \mathbf{T}(H)$.*

2. *Suppose $b, f \in H$, and $Y \subset S$ is a subsurface with $\xi(Y) \neq 3$, such that $b \swarrow^d Y$ and $Y \searrow^d f$. Then H contains a unique tight geodesic k such that $D(k) = Y$, $b \swarrow^d k$ and $k \searrow^d f$.*

3. *For every geodesic k in H other than g_H, there are $b, f \in H$ such that $b \swarrow^d k \searrow^d f$.*

We assume that all our hierarchies are **complete** in the terminology of [Min02], i.e. all subsurfaces of complexity ≥ 4 that can be the support of a geodesic in the hierarchy are exhausted in the hierarchy.

In Lemma 5.13 of [Min02] Minsky shows that **Hierarchies exist**, i.e. for any two markings $\mathbf{I}$ and $\mathbf{T}$ of S, there exists a hierarchy H with $\mathbf{I}(H) = \mathbf{I}$ and $\mathbf{T}(H) = T$.

Definition 3.6. *A* **slice** *of a hierarchy H is a set τ of pairs (h, v), where $h \in H$ and v is a simplex of h, satisfying the following properties:*

S1: A geodesic h appears in at most one pair in τ.

S2: There is a distinguished pair (h_τ, v_τ) in τ, called the bottom pair of τ. We call h_τ the bottom geodesic.

S3: For every $(k, w) \in \tau$ other than the bottom pair, $D(k)$ is a component domain of $(D(h), v)$ for some $(h, v) \in \tau$.

Note that **slices correspond to pants decompositions** in our context as we consider moves in subsurfaces of complexity ≥ 4. Elementary moves consist of replacing a single pants decomposition v by another w such that v and w agree on the complement of a complexity 4 subsurface W. Since W is either a 4-holed sphere $S_{0,4}$ or a one-holed torus $S_{1,1}$, an elementary move on W consists of replacing a simple closed curve $\alpha \subset W$ by another $\beta \subset W$ such that the intersection number of α and β is the minimal possible (two for $S_{0,4}$ and one for $S_{1,1}$).

Definition 3.7. *A pair (h, v) in τ is* forward movable *if:*

M1: v is not the last simplex of h. Let $v' = \text{succ}(v)$.

M2: For every $(k, w) \in \tau$ with $D(k) \subset D(h)$ and $v'|_{D(k)} \neq \emptyset$, w is the last simplex of k.

When this occurs we can obtain a slice τ' from τ by replacing (h, v) with (h, v'), erasing all the pairs (k, w) that appear in condition (M2), and inductively replacing them (starting with component domains of $(D(h), v')$) so that the final τ' satisfies

M2': For every $(k', w') \in \tau'$ with $D(k') \subset D(h)$ and $v|_{D(k')} \neq \emptyset$, w' is the first simplex of k'.

Then τ' exists and is uniquely determined by this rule. We write $\tau \to \tau'$, and say that the move advances *(h, v) to (h, v').*

We shall need the notion of a *resolution*, which is a sequence $\{\tau_i\}_{i=0}^N$ in $V(H)$, such that $\tau_i \to \tau_{i+1}$, In Lemma 5.7 of [Min02], Minsky proves that **resolutions exist**. In Lemma 5.8 of [Min02], Minsky proves that **resolutions sweep**. That is to say, if H is a hierarchy and $\{\tau_i\}_{i \in \mathcal{I}}$ a resolution, then for any pair (h, v) with $h \in H$ and v a simplex of h, there is a slice τ_i containing (h, v). Furthermore, if v is not the last simplex of h then there is exactly one elementary move $\tau_i \to \tau_{i+1}$ which advances (h, v).

Lemma 5.16 of Minsky [Min02]) shows that if $\{\tau_i\}_{i \in \mathcal{I}}$ be a resolution If v is a vertex in H, then $J(v) = \{i : (h, v) \in \tau_i \text{ for some } h\}$ is an interval in $\mathbb{Z}$.

3.2 The Model and the Bi-Lipschitz Model Theorem

In [Min02], Minsky constructs a model manifold M_ν associated to end-invariants ν. $M_\nu[0]$ denotes M_ν minus the collection of Margulis tubes and horoball neighborhoods of cusps. $M_\nu[0]$ is built up as a union of standard 'blocks' of a finite number of topological types as follows.

Minsky Blocks

Given a 4-edge e in H, let g be the 4-geodesic containing it, and let $D(e)$ be the domain $D(g)$. Let e^- and e^+ denote the initial and terminal vertices of e.

To each e a Minsky block $B(e)$ is assigned as as follows:

$$B(e) = (D(e) \times [-1,1]) \setminus (\text{ } \textbf{collar } (e^-) \times [-1,-1/2] \cup \textbf{collar } (e^+) \times (1/2,1]).$$

That is, $B(e)$ is the product $D(e) \times [-1,1]$, with solid-torus trenches dug out of its top and bottom boundaries, corresponding to the two vertices e^- and e^+ of e.

The *gluing boundary* of $B(e)$ is

$$\partial_\pm B(e) \equiv (D(e) \setminus \textbf{collar}(e^\pm)) \times \{\pm 1\}.$$

The gluing boundary is always a union of three-holed spheres. The rest of the boundary is a union of annuli. A model $M_\nu[0]$ is constructed in [Min02] by taking the disjoint union of all the Minsky blocks and identifying them along three-holed spheres in their gluing boundaries. The rule is that whenever two blocks B and B' have the same three-holed sphere Y appearing in both $\partial^+ B$ and $\partial^- B'$, these boundaries are identified using the identity on Y. The hierarchy serves to organize these gluings and insure that they are consistent.

Since the blocks are glued along $S_{0,3}$ components in the gluing boundary, a slice corresponds to a map of a pair of pants decomposition into M in the complement of Margulis tubes. We think of each component $S_{0,3}$ as horizontal. The collection of $S_{0,3}$'s corresponding to a slice is called a split level surface. Thus slices of the hierarchy correspond to **horizontal** split level surfaces. Two such slices related by an elementary move on the pants complex are related by a 4-geodesic supported in some W of complexity 4 and hence restricted to $W \times I$ form the top and bottom boundaries of a Minsky block. This forges a connection to the vertical annulus structure defined at the beginning of this section and is made explicit by the following theorem proven in [Min02].

Theorem 3.8. *(Theorem 8.1 of Minsky [Min02])* $M_\nu[0]$ *admits a proper flat orientation-preserving embedding* $\Psi : M_\nu[0] \to S \times \mathbb{R}$.

We shall be requiring the following **Bi-lipschitz Model Theorem** of Brock, Canary and Minsky [BCM04] for surface groups, which is the main Theorem of [BCM04]. In [BCM04], it is shown that the model manifold built in [Min02] is in fact bi-Lipschitz homeomorphic to the hyperbolic manifold with the same end-invariants.

Theorem 3.9. *Given a surface S of genus g, there exists L (depending only on g) such that for any doubly degenerate hyperbolic 3-manifold M without*

accidental parabolics homotopy equivalent to S, there exists an L-bi-Lipschitz map from M to the Minsky Model for M.

It is clear that if α be a curve on the boundary torus of a vertical Margulis tube T in the Minsky model bounding a totally geodesic disk in T, then the length of α is not less than the number of Minsky blocks abutting T. In fact, if the the number of Minsky blocks abutting T be n, then the length of the core geodesic of T is at most $O(\frac{1}{n})$. In particular we have the following Lemma.

Lemma 3.10. *Given $l > 0$ there exists $N \in \mathbb{N}$ such that the following holds. Let v be a vertex in the hierarchy H such that the length of the core curve of the Margulis tube T_v corresponding to v is greater than l. Then the number of Minsky blocks abutting T_v is at most N.*
Next suppose $(h, v) \in \tau_i$ for some i such that $h \searrow Y$, and D is a component of $Y \setminus v$. Also suppose that $h_1 \in H$ such that D is the support of h_1. Then the length of h_1 is at most N.

4   Split Geometry

For our purposes a simply or totally degenerate surface group will be a geometrically infinite surface group *without accidental parabolics.* The aim of this section is to extract a special sequence of split level surfaces from the Minsky model for a simply or totally degenerate manifold.

Fix an $l > 0$ (l will be less than the Margulis constant for hyperbolic 3-manifolds and determined by the **Drilling Theorem** to be used in the next subsection). We shall henceforth refer to Margulis tubes that have core curve of length $\leq l$ as **thin Margulis tubes** and the corresponding vertex v as a **thin vertex**.

4.1   Constructing Split Surfaces

We now proceed to construct a sequence of split surfaces exiting the end(s). For convenience start with a doubly degenerate surface group. Construct a complete hierarchy H following Minsky as in Lemma 5.13 of [Min02] (See previous Section). Let $\cdots, \tau_{i-1}, \tau_i, \tau_{i+1}, \cdots$ be a resolution and let S_i^s be the split level surface corresponding to the slice τ_i. Then each $J(v)$ (for v appearing in H) is an interval. Consider the family of intervals $\{J(v) : v \in g_H$, where g_H is the distinguished main geodesic (base geodesic) for the hierarchy H. Then $\bigcup_v \{J(v) : v \in g_H\} = \mathbb{Z}$. This follows from the fact that each τ_i, and hence each split level surface constructed has a simple closed curve corresponding to some vertex in g_H.

Definition 4.1. *A curve v in H is l-**thin** if the core curve of the Margulis tube T_v has length less than or equal to l.*
A curve v is said to split a pair of split surfaces S_i^s and S_j^s ($i < j$) if v occurs in both τ_i and τ_j.

*A pair of split surfaces S_i^s and S_j^s $(i < j)$ is said to be an l-**thin pair** if there exists an l-thin curve v such that v is a curve of both τ_i and τ_{j-1}.*

*A pair of split surfaces S_i^s and S_j^s $(i < j)$ is said to be an l-**thick pair** if no curve $v \in \tau_k$ is l-thin for $i < k < j$.*

An l-thick pair of split surfaces S_i^s and S_j^s $(i < j)$ is said to be (k, K)-separated if for all $x \in S_i^s$ and all $y \in S_j^s$ $k \leq d(x, y) \leq K$.

We drop l from l-thick or l-thin when it is understood.

Any pair v_i, v_{i+1} of simplices (multicurves) which form successive vertices of the base geodesic g_H are at a distance of 1 from each other by tightness of g_H. Now let τ_k denote the last slice in which (g_H, v_i) appears. Then τ_{k+1} is the first slice in which (g_H, v_{i+1}) appears by the choice of the resolution sequence $\{\tau_i\}$ (where the move from τ_k to τ_{k+1} occurs only when all slices where (g_h, v_i) occurs are exhausted).

Let τ_{n_i} be the first slice in the resolution such that $(g_H, v_i) \in \tau_{n_i}$ and Σ_i^s be the split surface corresponding to τ_{n_i}. Let Σ_{i+}^s be the split surface corresponding to $\tau_{n_{i+1}-1}$. The pair $(\Sigma_i^s, \Sigma_{i+}^s)$ is temporarily designated B_i^s.

Given the sequence of split surfaces S_i^s constructed, we first note that the simplices v_i on the base geodesic g_H give rise to (collections of) vertical annuli A_i (corresponding to multicurves v_i) that split B_i^s. However, since we have a threshold value l for core-curves, it is not necessary that any of the vertices of the simplex v_i corresponds to a thin Margulis tube. If some vertex in v_i is thin, then the corresponding Margulis tube is thin (by definition) and splits the thin pair B_i^s.

Interpolating Split Surfaces

We shall proceed to interpolate a sequence of surfaces between S_i^s and S_{i+1}^s if none of the vertices in the simplex v_i are thin. Assume therefore that none of the curves in v_i are thin. Then $S - v_i$ consists of a number of **component domains**. If D is such a component domain, there exists a geodesic g_D in the hierarchy whose domain is D and whose vertices occur in the resolution between Σ_i^s and Σ_{i+1}^s. Hence all simplices v_D of g_D are such that both (g_D, v_D) and (g_H, v_H) belong to the same slice τ of the given resolution. Further, the number of such component domains D is bounded uniformly in terms of the genus of the surface $((2g + 2)$ is always an upper bound).

Since none of the vertices in v_i are thin, then (fixing a D) g_D has length bounded by some uniform M_0 by Lemma 3.10. Let $v_{i1}, \cdots v_{ik}$ be the vertices on g_D. Then $k \leq M_0$. Construct the corresponding split surfaces. Repeat this for each such component domain. Let $S_{ij}^s, j = 1 \cdots n_i$ be the corresponding split surfaces constructed, renumbering them (if necessary) so that they occur in increasing order along the resolution. Then $n_i \leq M_0(2g + 2)$.

Again (iteratively) as before, two successive vertices on the same g_D give rise to the two possibilities mentioned above, viz. existence and non-existence of some thin curve $v \in g_D$ splitting successive split surfaces. If such a curve exists, the corresponding pair is a thin split pair, by definition. Else we proceed up the hierarchy.

Thus, in brief, we iterate the above construction as follows:

• If one meets a thin Margulis tube at some stage, STOP and go to a different component domain.
• Else continue iteratively up the hierarchy

Note that we are bound to stop in finite time as the hierarchy is of finite height determined purely by the number of nested component domains that may exist. This is determined by the genus of S. Let h_0 denote this height.

Pushing Split Surfaces Apart

If we do not encounter any thin Margulis tubes on the way in a sequence of iterations, then at the last step, we shall encounter a thick pair.

Note that in this process, each step of the iteration gives rise to an increase in the number of split surfaces by a multiplicative factor (n_i) bounded in terms of M_0 and the number of possible component domains, which in turn is bounded in terms of the genus g of the surface.

Thus, the number of split surfaces between S_i^s and S_{i+1}^s is bounded uniformly by some number N_0. Some of them may share a common subsurface, in which case, we push them apart by a small but definite amount, so that successive split surfaces are separated from each other by a definite amount as follows:

Note that each split surface is built out of a finite union of horizontal boundaries of Minsky blocks. By the construction of Minsky blocks, any such horizontal surface has a vertical product neighborhood of height $\frac{1}{2}$ away from splitting tubes. Thus pushing successive splitting surfaces apart by a uniform distance $= \frac{1}{3N_0}$, we would have moved a total height of at most $\frac{1}{3N_0} \times N_0 = \frac{1}{3}$ vertically along the product neighborhood. This would ensure that successive split surfaces are separated by at least $\frac{1}{3N_0}$.

To ensure smooth computation, we rescale the entire manifold by a multiplicative factor of $3N_0$, ensuring that successive split surfaces are separated by at least 1 away from splitting tubes.

Thick Block

Fix constants D, ϵ and let $\mu = [p, q]$ be an ϵ-thick Teichmuller geodesic of length less than D. μ is ϵ-thick means that for any $x \in \mu$ and any closed geodesic η in the hyperbolic surface S_x over x, the length of η is greater than ϵ. Now let B denote the universal curve over μ reparametrized such that the length of μ is covered in unit time. Thus $B = S \times [0, 1]$ topologically.

B is given the path metric and is called a **thick building block**.

Note that after acting by an element of the mapping class group, we might as well assume that μ lies in some given compact region of Teichmuller space. This is because the marking on $S \times \{0\}$ is not important, but rather its position relative to $S \times \{1\}$ Further, since we shall be constructing models only up to quasi-isometry, we might as well assume that $S \times \{0\}$ and $S \times \{1\}$ *lie in the orbit* under the mapping class group of some fixed base surface. Hence μ can be further simplified to be a Teichmuller geodesic joining a pair (p, q) amongst a finite set of points in the orbit of a fixed hyperbolic surface S.

Re-indexing: Re-index the specially selected split surfaces such that Σ_i^s and Σ_{i+1}^s are consecutive split surfaces amongst the split surfaces chosen.

Then one of the following occurs:

1) A thin curve splits the pair $(\Sigma_i^s, \Sigma_{i+1}^s)$ in which case $B^s = (\Sigma_i^s, \Sigma_{i+1}^s)$ is a **thin split pair**

2) $B^s = (\Sigma_i^s, \Sigma_{i+1}^s)$ is a **thick pair**. Note that in the above construction, any thick pair of split surfaces are connected to each other by a *uniformly* bounded number of moves. It follows that there exist (k, K) such that any thick pair constructed in the above sequence is automatically (k, K)-separated.

Lemma 4.2. *There exists n such that each thin curve splits at most n split surfaces in the above sequence.*

Proof: Since, for the initial sequence of split surfaces Σ_i^s (before re-indexing), two successive ones trap between them at most N_0 split surfaces (obtained by the iterative construction above), it suffices to prove that any thin curve splits a uniformly bounded number of these. If a curve v splits Σ_i^s and Σ_j^s, then we conclude that v corresponds to a curve in the pants decomposition of both Σ_i^s and Σ_j^s. Hence the distance between the corresponding curves v_i and v_j in the base geodesic g_H must be at most 2. By the construction of the S_i^s, this shows that v splits at most $2N_0$ split surfaces. Taking $n = 2N_0$, we are through. $\square$

We have thus constructed from the Minsky model the following:

Definition-Theorem 4.3. (WEAK SPLIT GEOMETRY)
1) A sequence of split surfaces S_i^s exiting the end(s) of M, where M is marked with a homeomorphism to $S \times J$ (J is $\mathbb{R}$ or $[0, \infty)$ according as M is totally or simply degenerate). $S_i^s \subset S \times \{i\}$.
2) A collection of Margulis tubes $\mathcal{T}$.
3) For each complementary annulus of S_i^s with core σ, there is a Margulis tube T whose core is freely homotopic to σ and such that T intersects the level i. (What this roughly means is that there is a T that contains the complementary annulus.) We say that T splits S_i^s.
*4) There exist constants $\epsilon_0 > 0, K_0 > 1$ such that for all i, either there exists a Margulis tube splitting both S_i^s and S_{i+1}^s, or else $S_i(= S_i^s)$ and $S_{i+1}(= S_{i+1}^s)$ have injectivity radius bounded below by ϵ_0 and bound a **thick block** B_i, where a thick block is defined to be a K_0 bi-Lipschitz homeomorphic image of $S \times I$.*
5) $T \cap S_i^s$ is either empty or consists of a pair of boundary components of S_i^s that are parallel in S_i.
6) There is a uniform upper bound $n = n(M)$ on the number of surfaces that T splits.
*A model manifold satisfying conditions (1)-(6) above is said to have **weak split geometry**.*

The statements about a thick block in Condition (4) above follows from the fact that a bounded number of moves of thick split surfaces corresponds roughly to a bounded element of the mapping class group, which in turn follows from the fact (see [MM00]) that the marking complex and the mapping class group are quasi-isometric.

4.2   Split Blocks

In this subsection, we shall 'fill' the regions between split surfaces. Note that from Condition (4) of Definition-Theorem 4.3 we have a prescription for 'filling' thick split pairs to form a thick block.

Topologically, an **extended split subsurface** S^s of a surface S is a (possibly disconnected, proper) subsurface with boundary such that
1) each component of S^s is an essential subsurface of S.
2) no component of S^s is an annulus.
3) $S - S^s$ consists of a non-empty family of non-homotopic essential annuli, none of which are homotopic into the boundary of S^s.

Geometrically, we assume that S is given some finite volume hyperbolic structure. An extended split subsurface S^s of S has bounded geometry, i.e.
1) each boundary component of S^s is of length ϵ_0, and is in fact a component of the boundary of $N_k(\gamma)$, where γ is a hyperbolic geodesic on S, and $N_k(\gamma)$ denotes its k-neighborhood.
2) For any closed geodesic β on S, either $\beta \subset S - S^s$, or, the length of any component of $\beta \cap (S - S^s)$ is greater than ϵ_0.

Topologically, a **split block** $B^s \subset B = S \times I$ is a topological product $S^s \times I$ for some *not necessarily connected* S^s. However, its upper and lower boundaries need not be $S^s \times 1$ and $S^s \times 0$. We only require that the upper and lower boundaries be *extended split subsurfaces* of S^s. This is to allow for Margulis tubes starting (or ending) within the split block. Such tubes would split one of the horizontal boundaries but not both. We shall call such tubes **hanging tubes**. Connected components of split blocks are called **split components**. We demand that there is a *non-empty* collection of Margulis tubes splitting a split block. However, re-iterating what has been mentioned above, we *do not* require that the upper (or lower) horizontal boundary of a split component K be connected. This happens due to the presence of *hanging tubes*. See figure below, where the left split component has four hanging tubes and the right split component has two hanging tubes. The vertical space between the components is the place where a Margulis tube splits the split block into two split components.

Note that the whole manifold M is then the union of
a) Thick blocks (homeomorphic to $S \times I$)
b) Split blocks (homeomorphic to $S^s \times I$ for some split surfaces)
c) Margulis tubes.

The union of thick blocks and split blocks give rise to the complement (in $M = S \times J$) of a special collection of Margulis tubes. Each of these Margulis tubes splits a uniformly bounded number of split blocks and might end in a hanging tube.
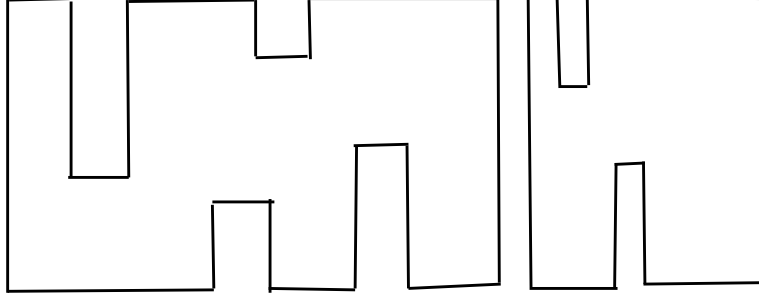
Figure 3: *Split Components of Split Block with hanging tubes*

4.3 Electrocutions

We define a **welded split block** to be a split block with identifications as follows: Components of $\partial S^s \times 0$ are glued together if and only if they correspond to the same geodesic in $S - S^s$. The same is done for components of $\partial S^s \times 1$. A simple closed curve that results from such an identification shall be called a **weld curve**. For hanging tubes, we also glue the boundary circles of their *lower or upper boundaries* by simply collapsing $S^1 \times [-\eta, \eta]$ to $S^1 \times \{0\}$. The same construction is repeated for all $i \geq 0$ by replacing $0, 1$ by $i, i+1$ respectively. For doubly degenerate groups we need to proceed in the negative direction too, from 0 to -1 and then inductively from $-i$ to $-(i+1)$.

Let the metric product $S^1 \times [0, 1]$ be called the **standard annulus** if each horizontal S^1 has length ϵ_0. For hanging tubes the standard annulus will be taken to be $S^1 \times [0, 1/2]$.

Next, we require another pseudometric on B which we shall term the **tube-electrocuted metric**. We first define a map from each boundary annulus $S^1 \times I$ (or $S^1 \times [0, 1/2]$ for hanging annuli) to the corresponding standard annulus that is affine on the second factor and an isometry on the first. Now glue the mapping cylinder of this map to the boundary component. The resulting 'split block' has a number of standard annuli as its boundary components. Gluing the standard annuli corresponding to the same Margulis tube together by the 'identity' map, we obtain the *welded (or stabilized) split block* B^{st} and the resulting metric on B^{st} is the **weld-metric** d_{wel}.

Glue boundary components of B^{st} corresponding to the same geodesic together to get the **tube electrocuted metric** on B as follows. Suppose that two boundary components of B^{st} correspond to the same geodesic γ. In this case, these boundary components are both of the form $S^1 \times I$ or $S^1 \times [0, \frac{1}{2}]$ where there is a projection onto the horizontal S^1 factor corresponding to γ. Let $S_l^1 \times J$ and $S_r^1 \times J$ denote these two boundary components (where J denotes I or $[0, \frac{1}{2}]$). Then each $S^1 \times \{x\}$ has length ϵ_0. Glue $S_l^1 \times J$ to $S_r^1 \times J$ by the natural 'identity map'. Finally, on each resulting $S^1 \times \{x\}$ put the zero metric. Thus the annulus $S^1 \times J$ obtained via this identification has the zero metric in the *horizontal direction* $S^1 \times \{x\}$ and the Euclidean metric in the *vertical direction* J. The resulting block will be called the **tube-electrocuted block** B_{tel}

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and the pseudometric on it will be denoted as d_{tel}. Note that B_{tel} is homeomorphic to $S \times I$. The operation of obtaining a *tube electrocuted block and metric* (B_{tel}, d_{tel}) from a split block B^s shall be called *tube electrocution*. Note that a tube electrocuted block or a welded block is homeomorphic to $S \times I$. Note also that d_{tel} is obtained from d_{wel} by putting the zero metric on the horizontal circles of length ϵ_0 in the standard annulus.

A lift of a split component to the universal cover of the block $B = S \times I$ or $B_{tel} = S \times I$ shall be termed a **split component** $\widetilde{K}$ of $\widetilde{B}$ or $\widetilde{B_{tel}}$. There are lifts of splitting Margulis tube that share the boundary of a lift $\widetilde{K}$ in $\widetilde{M}$. Adjoining these lifts to $\widetilde{K}$ we obtain **extended split components**.

Also, let d_G be the (pseudo)-metric obtained by electrocuting the collection of split components $\widetilde{K}$ in $\widetilde{B_{tel}}$. d_G will be called the the **graph metric**.

Definition 4.4. *Let $CH(K)$ denote the convex hull of an extended split component $\widetilde{K}$ in $\widetilde{M}$. $\widetilde{K}$ is said to be D_0-graph quasiconvex if the diameter $dia_G(CH(K))$ of $CH(K)$ in the graph metric d_G is bounded by D_0.*

Important Identification: The usual hyperbolic space $(\widetilde{M}, d)$ with lifts of *extended split components* electrocuted is basically the same as the space $(\widetilde{M}, d_{wel})$ with lifts of split components electrocuted. This is because $(\widetilde{M}, d)$ and $(\widetilde{M}, d_{wel})$ agree outside the collection of Margulis tubes $\mathcal{T}$. Further, the identity map from $(\widetilde{M} \setminus \mathcal{T}, d)$ to $(\widetilde{M} \setminus \mathcal{T}, d_{wel})$ extends naturally to a map from $(\widetilde{M}, d)$ to $(\widetilde{M}, d_{wel})$ by mapping Margulis tubes to welded standard annuli (by a homotopy equivalence of the solid Margulis tube to a standard annulus if one likes). Thus, the graph metric on $(\widetilde{M}, d)$ with lifts of *extended split components* electrocuted is quasi-isometric to the graph metric on $(\widetilde{M}, d_{wel})$ with lifts of split components electrocuted. We shall henceforth **not** distinguish between these two graph metrics.

4.4 Quasiconvexity of Split Components

We now proceed to show further that split components are (not necessarily uniformly) quasiconvex in the hyperbolic metric, and uniformly quasiconvex in the graph metric, i.e. we require to show *hyperbolic quasiconvexity* and *uniform graph quasiconvexity* of (extended) split components.

Hyperbolic Quasiconvexity:

We shall specialize the Thurston-Canary covering theorem given below, [Thu80] [Can96] to the case under consideration, viz. infinite index free subgroups of surface Kleinian groups.

Theorem 4.5. Covering Theorem *[Thu80] [Can96] Let $M = \mathbf{H}^3/\Gamma$ be a complete hyperbolic 3-manifold. A finitely generated subgroup Γ' is geometrically infinite if and only if it contains a finite index subgroup of a geometrically infinite peripheral subgroup.*

Let $CH(K)$ denote the convex hull of an (extended) split component $\widetilde{K}$. The subgroup corresponding to $\pi_1(K)$ takes the place of Γ' (in Theorem 4.5 above), which, being of infinite index in $\pi_1(S)$ cannot contain a finite index subgroup of $\pi_1(S)$ (the peripheral subgroup in this case). Theorem 4.5 then forces $\pi_1(M_1)$ to be geometrically finite and we have thus shown:

Lemma 4.6. *Given a split (or extended split) component K, there exists C_0 such that K is C_0-quasiconvex in M, i.e. any geodesic with end points in K homotopic into K lies in a C_0 neighborhood of K. Further, if $\tilde{K}$ is a copy of the universal cover of K in $\tilde{M}$, then the convex hull of $\tilde{K}$ lies in a C_0-neighborhood of $\tilde{K}$.*

Graph Quasiconvexity:
Next, we shall prove that each split component is uniformly graph quasiconvex. We begin with the following Lemma. Recall that we are dealing with simply or totally degenerate groups without accidental parabolics.

Lemma 4.7. *Let S_1 be a component of an extended split subsurface S_i^s of S. Any (non-peripheral) simple closed curve in S appearing in the hierarchy whose free homotopy class has a representative lying in S_1 must have a geodesic representative in M lying within a uniformly bounded distance of S_i^s in the graph metric.*

Proof : Suppose a curve v in the hierarchy is homotopic into S_1. Then v is at a distance of 1 (in the curve complex) from each of the boundary components of S_1. Let α be such a boundary component. Next, suppose the geodesic representative (in $\widetilde{M}$) of v intersects some block B_j^s. Then v must be at a distance of at most one from a base curve σ, i.e. a curve in the base geodesic g_H forming an element of the pants decomposition of the split surface S_j^s. By tightness, the distance from α to σ in the curve complex is at most 2. Hence the distance of S_j^s from S_i^s is $\leq 2n$ from Lemma 4.2. Therefore v is realized within a distance $2n$ of S_i^s in the graph metric. $\square$

Next, we show that any (non-peripheral) simple closed curve v_i in Σ (not just hierarchy curves as in Lemma 4.7) must be realized within a uniformly bounded distance in the graph metric. In fact we shall show further that any pleated surface which contains at least one boundary geodesic of Σ in its pleating locus lies within a uniformly bounded distance of S_i^s in the graph metric.

Choose a curve v_i homotopic to a simple closed curve on Σ. Let α denote its geodesic realization in the 3-manifold (i.e. α is the geodesic representative in the free homotopy class).

There exists a pleated (sub)surface Σ_p whose pleating locus contains v_i and whose boundary coincides with the geodesics representing the boundary components of Σ. Then Σ_p has bounded area by Gauss-Bonnet.

For the rest of the argument for Lemma 4.8 below, we need to assume only that Σ_p is a pleated surface with at least one boundary component coinciding with a geodesic representative of a component of $\partial\Sigma$. (The previous paragraph

just asserts that we have such a pleated surface with pleating locus containing v_i.)

Any hyperbolic surface S_h of genus bounded by the genus of S has *uniformly bounded diameter modulo ϵ-thin parts* for any ϵ. What this means is the following. Given any sufficiently small ϵ, let S_ϵ denote the set of points in S_h where injectivity radius is less than ϵ. Then by the Margulis lemma, S_ϵ consists of a disjoint union of thin annuli and cusps. Put the zero metric (electrocute) on each of these components to obtain S_{he}. Then the diameter of S_{he} is bounded in terms of ϵ.

Let us now return to Σ_p. By the Margulis lemma again, there exists ϵ_0 such that any point $z \in \Sigma_p \subset M$ with injectivity radius less than ϵ_0 lies within a Margulis tube in M. Further any such Margulis tube T_u corresponds to a curve u in the curve complex at distance *one* from v_i. Let $x \in \Sigma_p$. From the previous paragraph, we obtain a constant c (depending only on the genus of S and the Margulis constant), such that

1. either there exists a Margulis tube T_u such that $d(x, T_u) \leq c$ in the hyperbolic metric. Here the Margulis tube T_u corresponds to a curve u at distance 1 from v_i.

2. or, $d(x, y) \leq c$ for all $y \in \Sigma_p$. (These two are cases are not mutually exclusive.)

Since split surfaces are separated from each other, by a fixed amount, it follows that there exists K_c such that x lies at a distance of at most K_c from either T_u or some boundary curve γ in the graph metric. Therefore, by the triangle inequality and Lemma 4.7, for any $x \in S_i^s$, we have $d_G(x, \gamma) \leq K_c + 2n$, where d_G denotes the graph metric and n (in Lemma 4.7) depends only on the split geometry model. In particular the realization α must lie within a distance $K_c + 2n + 1$ of S_i^s in the graph metric. We have thus shown:

Lemma 4.8. *There exists $B > 0$ such that the following holds:*
Let Σ be a split subsurface of S_i^s. Then any pleated surface with at least one boundary component coinciding with a geodesic representative of a component of $\partial \Sigma$ must lie within a B-neighborhood of S_i^s in the graph metric. In particular, every simple closed curve in S homotopic into Σ has a geodesic representative within a B-neighborhood of S_i^s in the graph metric.

Remark: In [Bow05b], Bowditch indicates a method to obtain a related (stronger) result that given $B_1 > 0$, there exists $B_2 > 0$ such that any two simple closed curves realized within a Hausdorff distance B_1 of each other in M are within a distance B_2 of each other in the curve complex.

4.5   Proof of Uniform Graph-Quasiconvexity

We need to prove the uniform graph quasiconvexity of split components.
Let B^s be a split block with a Margulis tube T. We aim at showing:

Proposition 4.9. Uniform Graph Quasiconvexity of Split Components:
Each component of $B^s - T$ is uniformly (independent of B^s) graph-quasiconvex in the model manifold M.

The proof of Proposition 4.9 will occupy the rest of this subsection.

To go about proving Proposition 4.9 above, we first recall that $B^s \subset B = S \times I$ is a split block. $(B - B^s) \subset \bigcup_i T_i$ for a finite collection of (solid) Margulis tubes T_i. Let K be a component (hence by definition, a *split component*) of $(B - B^s)$. Then $K = (S_1 \times I)$ topologically for a subsurface S_1 of S. Also, $\partial K = \partial S_1 \times I$. Let $\partial S_1 = \bigcup_i \sigma_i = \sigma$ for a finite collection of curves σ_i. σ is thus a **multicurve**. Each σ_i is homotopic to the core curve of a Margulis tube T_i. Let $\mathbf{T} = \bigcup_i T_i$. $\mathbf{T}$ will be referred to as a **multi-Margulis tube**.

We have already shown (Lemma 4.6) that $\pi_1(S_1) \subset \pi_1(S)$ gives rise to a geometrically finite subgroup of $PSl_2(C)$. Let M_1 be the cover of M corresponding to $\pi_1(K) = \pi_1(S_1)$. Then M_1 is geometrically finite. Let $\mathbf{T}^1$ be the multi-Margulis tube in M_1 that consists of tubes that are (individually) isometric to individual components of the multi-Margulis tube $\mathbf{T} \subset K \subset M$.

The Drilled Manifold

Let M_{1d} be the hyperbolic manifold obtained from M_1 by drilling out the core curves of $\mathbf{T}$. We remark here (following Brock-Bromberg [BB04]) that the drilled manifold is the unique hyperbolic manifold which has the same conformal structure on its domain of discontinuity, but has core curves of $\mathbf{T}$ corresponding to rank 2 parabolics. Since M_1 is geometrically finite, so is M_{1d}.

We first observe that the boundary of the augmented Scott core X of M_{1d} is incompressible away from cusps. To see this, note that X is double covered by a copy of $D \times I$ with solid tori drilled out of it, where D is the *double* of S_1 (obtained by doubling S_1 along its boundary circles).

We introduce geometry and identify X with the convex core $CC(M_{1d})$ of M_{1d}. We also identify D with the convex core boundary. Since D is incompressible away from cusps, we conclude from a theorem of Thurston:

Lemma 4.10. *[Thu80] D is a pleated surface.*

Next, since M_1 is the cover of M corresponding to $\pi_1(K) \subset \pi_1(M)$, K lifts to an embedding into M_1. Adjoin the multi-Margulis tube $\mathbf{T}^1$ to (the lifted) K to get an augmented split component K_1. Let $K_{1d} \subset M_{1d}$ denote K_1 with the components of $\mathbf{T}^1$ drilled. We want to show that D lies within a uniformly bounded distance of K_{1d} in the lifted graph metric on M_{1d}. This would be enough to prove a version of Proposition 4.9 for the drilled manifold M_{1d} as the split geometry structure gives rise to a graph metric on M, hence a graph metric on M_1 and hence again, a graph metric on M_{1d}. Finally, we shall use the Drilling Theorem of Brock-Bromberg [BB04] to complete the proof of Proposition 4.9.

Lemma 4.11. *There exists C_1 such that for any split component K, D lies within a uniformly bounded neighborhood of K_{1d} in M_{1d}.*

Proof: Case 1: $D \cap K_{1d} \neq \emptyset$
If D intersects K_{1d}, then as in Lemma 4.8, D lies within a uniformly bounded

neighborhood of K_{1d} in the graph-metric. (Recalling briefly: Since the genus of D is less than twice the genus g of S, its area is uniformly bounded. Hence, its diameter modulo thin parts is uniformly bounded. This gives the required conclusion. For details we refer back to Lemma 4.8.)

Case 2: $D \cap K_{1d} = \emptyset$

This is the more difficult case because *a priori* D might lie far from K_{1d}. Let B denote the block (in the split geometry model of M) containing K. Let $B_1 \subset M_1$ denote its cover in M_1. Let B_{1d} denote B_1 with $\mathbf{T}^1$ drilled.

Then $B_{1d} - K_{1d}$ is topologically a disjoint union of 'vertically thickened flaring annuli' A_i. Each A_i is of the form $S^1 \times [0, \infty)$ where $S^1 \times \{0\}$ lies on T_i.

What this means is the following. Identifying B with $S \times I$, we may identify B_1 with $S_1^a \times I$, where S_1^a is the cover of S corresponding to the subgroup $\pi_1(S_1) \subset \pi_1(S)$. Then S_1^a may be regarded as S_1 union a finite collection of flaring annuli A_i (one for each boundary component of S_1). Thus B_1 is the union of a core K_1 and a collection of *vertically thickened flaring annuli* of the form $A_i \times I$. Hence B_{1d} is the union of a core K_{1d} and the collection of *vertically thickened flaring annuli* $A_i \times I$. Also the boundary $\partial A_i = A_i \cap T_i$ is a curve of fixed length ϵ_0. Let us fix one such annulus A_1. Refer figure below:
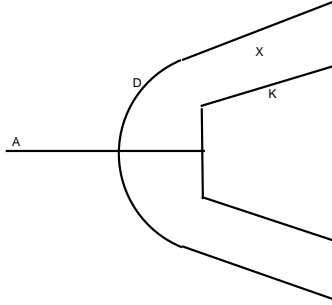

Figure 4: *Graph Quasiconvexity*

Since D bounds X and X contains K_{1d}, D must intersect A_1 in an essential loop α parallel to ∂A_1. Hence D must contain an annulus of the form $\alpha \times I \subset A_1 \times I \subset B_{1d}$. Since the I-direction here is 'vertical', the length of I is at least h_0, the uniform lower bound on the height of split blocks B^s. Hence for at least some $t \in I$, the length of $\alpha \times \{t\}$ is uniformly bounded (by $\frac{2\pi(4g-4)}{h_0}$). Much more is true in fact, but this is enough for our purposes.

Since $\alpha \times \{t\} \subset A_1 \times \{t\}$ and the latter is a flaring annulus, it follows that there is some point $p \in \alpha \times \{t\} \subset D$ such that $d(p, \alpha)$ is uniformly bounded (in terms of the genus of S and the minimal height of split blocks h_0).

Again, as in the proof of Lemma 4.8, the diameter of D is uniformly bounded in the graph metric lifted to M_{1d}. Hence, by the triangle inequality, D lies in a uniformly bounded neighborhood of K in the graph metric. $\square$

An Alternate Proof of Lemma 4.11: A simpler proof of the fact that D lies in a uniformly bounded neighborhood of K_1 in the graph metric may

alternately be obtained directly as follows. First, that M_1 is geometrically finite by the **Covering Theorem** of Thurston [Thu80] and Canary [Can96] (See Lemma 4.6). Next, by a theorem of Canary and Minsky [CM96], it follows that the convex hull boundary D of M_1 can be approximated by simplicial hyperbolic surfaces homotopic to D with short tracks. Thus any simplicial hyperbolic approximant D_a would have to have bounded area and hence bounded diameter modulo Margulis tubes (as in Lemma 4.8). Hence so would D. Now, we repeat the argument in the proof of Lemma 4.14, to conclude that D and hence the convex core $CC(M_1)$ of M_1 lies in a uniformly bounded neighborhood of K_1 in the graph metric. This approach would circumvent the use of the Drilling Theorem at this stage. However, since we shall again need it below, we retain our approach here.

Since D bounds X, we would like to claim that the conclusion of Lemma 4.11 follows with X in place of D. Though this does not a priori follow in the hyperbolic metric, it does follow for the graph metric. This is because the double cover of X is a 'drilled quasifuchsian' manifold (i.e. it is essentially $(D \times I)$ with some short curves drilled). Further, any point in the convex core of a quasifuchsian $(D \times I)$ is close to a simplicial hyperbolic surface by a *Filling Theorem* of Thurston [Thu80], Ch. 9.5 (generalized by Canary in [Can96]). Essentially the same argument as in Lemma 4.11 applies now. Details will be given below.

The difference between the drilled manifold and a quasifuchsian $(D \times I)$ is that the drilled manifold may be realized as a geometric limit of quasifuchsian manifolds (as in [KT90]). By the Drilling Theorem (see below) the complement of cusps in the drilled manifold and the complement of Margulis tubes in the quasifuchsian manifold are uniformly bi-Lipschitz. This allows us to pass back and forth between the drilled and 'undrilled' manifolds.

The **Drilling Theorem** of Brock and Bromberg [BB04], which built on work of Hodgson and Kerckhoff [HK98] [HK05] is given below. We invoke a version of this theorem which is closely related to one used by Brock and Souto in [BS06].

Theorem 4.12. *[BB04] For each $L > 1$, and n a positive integer, there is an $\ell > 0$ so that if M be a geometrically finite hyperbolic 3-manifold and $c_1, \cdots c_n$ are geodesics in M with length $\ell_M(c_i) < \ell$ for all c_i, then there is an L-bi-Lipschitz diffeomorphism of pairs*

$$h \colon (M \setminus \cup_i \mathbb{T}(c_i), \cup_i \partial \mathbb{T}(c)) \to (M_0 \setminus \cup_i \mathbb{P}(c_i), \cup_i \partial \mathbb{P}(c_i))$$

where $M \setminus \cup_i \mathbb{T}(c_i)$ denotes the complement of a standard tubular neighborhood of $\cup_i c_i$ in M, M_0 denotes the complete hyperbolic structure on $M \setminus \cup_i c_i$, and $\mathbb{P}(c_i)$ denotes a standard rank-2 cusp corresponding to c_i.

The *Filling Theorem* of Thurston [Thu80] (generalized by Canary [Can96]) we require is stated below.

Theorem 4.13. *[Thu80] [Can96] Given any quasifuchsian surface group Γ and $M = \mathbf{H}^3/\Gamma$ there exists $\delta > 0$ depending only on the genus of the surface such*

that for all $x \in CC(M)$, the convex core of M, there exists a simplicial hyperbolic surface Σ such that $d(x, \Sigma) \leq \delta$.

X is double covered by $D \times I$ with cores of some Margulis tubes drilled. Let X_1 denote this double cover. Note that X_1 is convex (being a double cover of the convex compact X). Perform Dehn filling on X_1 with sufficiently large coefficients to obtain a filled manifold X_{1f}. Then by Theorem 4.12, X_{1f} is uniformly quasiconvex in $M_{1f} = \mathbf{H}^3/\Gamma$ where Γ is a quasiFuchsian surface group obtained by the above Dehn filling. (Theorem 4.12 gives a uniform bi-Lipschitz map outside Margulis tubes, which in turn are contained within the convex core.)

Next, by Theorem 4.13, for all $x \in X_{1f}$ there exists a simplicial hyperbolic surface $\Sigma \subset X_{1f}$ such that $d(x, \Sigma) \leq \delta$ where δ depends only on the genus of D.

Returning to X_1 via the Drilling Theorem 4.12 we see that for all $x \in X_1$,

1. Either there exists a uniformly bi-Lipschitz image of a hyperbolic surface $\Sigma_1 \subset X_1$ such that $d(x, \Sigma_1) \leq \delta$ (if the simplicial hyperbolic Σ misses all filled Margulis tubes).

2. Or, there exists a uniformly bi-Lipschitz image of a subsurface Σ_1 of a hyperbolic surface such that $d(x, \Sigma_1) \leq \delta$ and such that the boundary of Σ_1 lies on a Margulis tube. (if the simplicial hyperbolic Σ meets some filled Margulis tubes. Here, we can take Σ_1 to be the image of the component of *(Σ minus Margulis tubes)* that lies near x).

Again, passing down to X under the double cover (from X_1 to X), we have, for all $x \in X$,

1. Either there exists a uniformly bi-Lipschitz image of a hyperbolic surface $\Sigma_1 \subset X$ parallel to D.

2. Or, there exists a uniformly bi-Lipschitz image of a subsurface Σ_1 of a hyperbolic surface such that $d(x, \Sigma_1) \leq \delta$ and such that the boundary of Σ_1 lies on a Margulis tube.

In either case, the argument for Lemma 4.11 shows that for all $x \in X$ the distance $d_G(x, K_{1d})$ is uniformly bounded (in the graph-metric d_G). Thus, we have shown that K_{1d} is uniformly graph-quasiconvex in M_{1d}.

Lemma 4.14. *There exists C_1 such that for any split component K, K_{1d} is uniformly graph-quasiconvex in M_{1d}.*

To complete the proof of Proposition 4.9 it is necessary to translate the content of Lemma 4.14 to the 'undrilled' manifold M_1. We shall need to invoke the Drilling Theorem 4.12 again.

Concluding the Proof of Proposition 4.9:

While recovering data about M_1, it is slightly easier to handle the case where $D \cap K_{1d} = \emptyset$.

Case 1: $D \cap K_{1d} = \emptyset$

Filling M_{1d} along the (drilled) $\mathbf{T}^1$, we get back M_1. Since D misses K_{1d}, the filled image of X in M_1 is C_1-quasiconvex for some C_1, depending on the bi-Lipschitz constant of Theorem 4.12 above. (One can see this easily for instance from the fact that there is a uniform Lipschitz retract of $M_{1d} - X$ onto D).

Case 2: $D \cap K_{1d} \neq \emptyset$

If D meets some Margulis tubes, we enlarge D to D' by letting D' be the boundary of $X_1 = X \cup \mathbf{T}^1$. The annular intersections of D with Margulis tubes are replaced by boundary annuli contained in the boundary of $\mathbf{T}$.

It is easy enough to check that the resulting augmented convex core X_1 is uniformly quasiconvex in the hyperbolic metric. To see this, look at a universal cover $\tilde{X}_1$ of X_1 in $\tilde{M}_{1d}$. Then $\tilde{X}_1$ is a union of $\tilde{X}$ and the lifts of T that intersect it. All these lifts of T are disjoint. Hence $\tilde{X}_1$ is a 'star' of convex sets all of which intersect the convex set $\tilde{X}$. By (Gromov) δ-hyperbolicity, such a set is uniformly quasiconvex.

Then as before, there is a uniform Lipschitz retract of $M_{1d} - X_1$ onto D'. But now D' misses the interior of K_{1d} and we can apply the previous argument.

By Theorem 4.12 above, the diameter of D (or D' if D intersects some Margulis tubes) in M_1 is bounded in terms of the diameter of D in M_{1d} and the **uniform** bi-Lipschitz constant L obtained from Theorem 4.12 above. Further, the distance of D from $K_1 \cup \mathbf{T}^1$ in M_1 is bounded in terms of the distance of D from $K_{1d} \cup \partial \mathbf{T}^1$ in M_{1d} and the bi-Lipschitz constant L.

Hence we can translate the content of Lemma 4.14 to the 'undrilled' manifold M_1. This concludes the proof of Proposition 4.9:

Split components are uniformly graph-quasiconvex. $\square$

Remark 1: Our proof above uses the fact that the convex core X of M_{1d} is a rather well-understood object, viz. a manifold double covered by a drilled convex hull of a quasi-Fuchsian group. Hence, it follows that the convex core X is uniformly congested, i.e. it has a uniform upper bound on its injectivity radius. This is an approach to a conjecture of McMullen [Bie] (See also Fan [Fan99a] [Fan99b]).

Remark 2: We implicitly use here the idea of drilling *disk-busting curves* introduced by Canary in [Can93] and used again by Agol in his resolution of the tameness conjecture [Ago04].

Remark 4.15. *Recall that extended split components were defined in $\widetilde{M}$ by adjoining Margulis tubes abutting lifts of split components to $\widetilde{M}$. The proof of Proposition 4.9 establishes also the uniform graph-quasiconvexity of extended split components in $\widetilde{M}$. The metric obtained by electrocuting the family of convex hulls of extended split components in $\widetilde{M}$ will be denoted as d_{CH}.*

4.6 Hyperbolicity in the graph metric

First a word about the modifications necessary for Simply Degenerate Groups.

Simply Degenerate Groups We have so far assumed, for ease of exposition, that we are dealing with totally degenerate groups. In a simply degenerate M, the Minsky model is *uniformly* bi-Lipschitz to M only in a neighborhood E of the end. In this case $(M \setminus E)$ is homeomorphic to $S \times I$. We declare $(M \setminus E)$ to be the first block - a 'thick block' in the split geometry model. Thus the boundary blocks of Minsky are put together to form one initial thick block. This changes the bi-Lipschitz constant, but the rest of the discussion, including Proposition 4.9 go through as before.

Construct a second auxiliary metric $\widetilde{M_2} = (\widetilde{M}, d_{CH})$ by electrocuting the elements $CH(K)$ of convex hulls of *extended split components*. We show that the spaces $\widetilde{M_1} = (\widetilde{M}, d_G)$ and $\tilde{M_2} = (\widetilde{M}, d_{CH})$ are quasi-isometric. In fact we show that the 'inclusion' map on the underlying set is a quasi-isometry. Note that the underlying metric on $\widetilde{M_1}$ before electrocution is $(\widetilde{M}, d_{wel})$, whereas the underlying metric on $\widetilde{M_2}$ before electrocution is the ordinary hyperbolic metric $(\widetilde{M}, d)$. There is a slight amount of ambiguity in the inclusion map. We demand only that the standard annulus (to which the two vertical boundaries of the associated Margulis tube is glued) is mapped within the Margulis tube. In the complement of the interior of Margulis tubes, the inclusion map is the identity.

Lemma 4.16. *The inclusion map on the underlying set M from M_1 to M_2 induces a quasi-isometry of universal covers $\widetilde{M_1}$ and $\widetilde{M_2}$.*

Proof: Let d_1, d_2 denote the electric metrics on $\widetilde{M_1}$ and $\widetilde{M_2}$. Since $K \subset CH(K)$ for every split component, we have right off

$$d_1(x, y) \leq d_2(x, y) \text{ for all } x, y \in \widetilde{M}$$

To prove a reverse inequality with appropriate constants, it is enough to show that each set $CH(K)$ (of diameter one in M_2) has uniformly bounded diameter in M_1. To see this, note that by definition of graph-quasiconvexity, there exists n such that for all K and each point a in $CH(K)$, there exists a point $b \in K$ with $d_1(x, y) \leq n$. Hence by the triangle inequality,

$$d_2(x, y) \leq 2n + 1 \text{ for all } x, y \in \widetilde{CH(K)}$$

Therefore,

$$d_2(x, y) \leq (2n + 1)d_1(x, y) \text{ for all } x, y \in \tilde{M}$$

This proves the Lemma. □

Remark 4.17. *By Lemma 2.1, $\widetilde{M_2} = (\widetilde{M}, d_{CH})$ is a hyperbolic metric space. By quasi-isometry invariance of Gromov hyperbolicity, so is $\widetilde{M_1} = (\widetilde{M}, d_G)$.*

Remark 4.18. *Note that the underlying sets for $\widetilde{M_1} = (\widetilde{M}, d_G)$ and $\widetilde{M_2} = (\widetilde{M}, d_{CH})$ are homeomorphic as topological spaces. Also, $\widetilde{M_1}$ is obtained by tube-electrocuting the welded metric, i.e. $(\widetilde{M}, d_{wel})$, whereas $\widetilde{M_2}$ is obtained by electrocuting the hyperbolic metric, i.e. $(\widetilde{M}, d)$ Note that the metrics $(\widetilde{M}, d_{wel})$ and $(\widetilde{M}, d)$ locally coincide off Margulis tubes. We need to set up a correspondence between paths in $(\widetilde{M}, d_{wel})$ and $(\widetilde{M}, d)$, and hence between $\widetilde{M_1} = (\widetilde{M}, d_G)$ and $\widetilde{M_2} = (\widetilde{M}, d_{CH})$. Paths $\alpha_i \subset \widetilde{M_i}$ are said to* **correspond** *if*
1) They coincide off Margulis tubes
2) Each piece of α_2 inside a (closed) Margulis tube is a geodesic in the hyperbolic metric d.
We shall not have need for this correspondence till Section 6.4, and to avoid clumsy notation, shall continue to refer to the underlying topological manifold of $\widetilde{M_1} = (\widetilde{M}, d_G)$ as $\widetilde{M}$.

We have thus constructed a sequence of split surfaces that satisfy the following two conditions in addition to Conditions (1)-(6) of Definition-Theorem 4.3 for the Minsky model of a simply or totally degenerate surface group:

Definition-Theorem 4.19. (SPLIT GEOMETRY)
7) Each split component $\widetilde{K} \subset \widetilde{B_i} \subset \widetilde{M}$ is (not necessarily uniformly) quasiconvex in the hyperbolic metric on $(\widetilde{M}, d_{wel})$.
8) Equip $\widetilde{M}$ with the graph-metric d_G obtained by electrocuting each split component $\widetilde{K}$. Then the convex hull $CH(\widetilde{K})$ of any split component $\widetilde{K}$ has uniformly bounded diameter in the metric d_G. We say that the components $\widetilde{K}$ are **uniformly graph-quasiconvex**. *It follows that $(\widetilde{M}, d_G)$ is a hyperbolic metric space.*
A model manifold satisfying conditions (1)-(8) above is said to have **split geometry**.

Combining the bi-Lipschitz model Theorem 3.9 of Brock-Canary-Minsky with Definition-Theorem 4.19 above we have the following.

Theorem 4.20. *Any simply or doubly degenerate surface group without accidental parabolics is bi-Lipschitz homeomorphic to a model of split geometry.*

5 Constructing Quasiconvex Ladders and Quasigeodesics

To avoid confusion we summarize the various metrics on $\widetilde{M}$ that will be used:
1) The hyperbolic metric d.
2) The weld-metric d_{wel} obtained after welding the boundaries of Margulis tubes to standard annuli (and before tube electrocution) where each horizontal circle of a Margulis tube T has a fixed non-zero length.

3) The tube-electrocuted metric d_{tel}.

4) The graph metric d_G.

There will be two (families of) metrics on the universal cover $\widetilde{S}$ of S:

1) The graph-electrocuted metric d_{Gel} obtained by electrocuting the amalgamation components of $\widetilde{S}$ that the lift of a weld-curve cuts $\widetilde{S}$ into.

2) The hyperbolic metric d on $\widetilde{S}$ obtained by lifting the metric on the *welded* surface. The term 'hyperbolic' is a slight misuse as the metric on S is obtained by cutting out thin annuli and then welding the boundaries of the resulting extended split surface together.

Note that the path metric induced on $\widetilde{S} \subset \widetilde{B}$ for B a split block is precisely d_{Gel}.

5.1   Construction of Quasiconvex Sets for Building Blocks

In this subsection, we describe the construction of a **hyperbolic ladder** $\mathcal{L}_\lambda$ restricted to building blocks B. Putting these together we will show later that $\mathcal{L}_\lambda$ is quasiconvex in $(\widetilde{M}, d_G)$.

Construction of $\mathcal{L}_\lambda(B)$ - Thick Block

Let B be a thick block. By definition B can be thought of as a universal curve over a Teichmuller geodesic $[\alpha, \beta]$. Let S_α denote the hyperbolic surface over α and S_β denote the hyperbolic surface over β.

Let $\lambda = [a, b]$ be a geodesic segment in $\widetilde{S}$. Let λ_{B0} denote $\lambda \times \{0\}$.

Let ψ be the lift of the 'identity' map from $\widetilde{S_\alpha}$ to $\widetilde{S_\beta}$. Let Ψ denote the induced map on geodesics and let $\Psi(\lambda)$ denote the hyperbolic geodesic joining $\psi(a), \psi(b)$. Let λ_{B1} denote $\Psi(\lambda) \times \{1\}$.

For the universal cover $\widetilde{B}$ of the thick block B, define

$$\mathcal{L}_\lambda(B) = \bigcup_{i=0,1} \lambda_{Bi}$$

Definition: Each $\widetilde{S} \times i$ for $i = 0, 1$ will be called a **horizontal sheet** of $\widetilde{B}$ when B is a thick block.

Construction of $\mathcal{L}_\lambda(B)$ - Split Block

As above, let $\lambda = [a, b]$ be a geodesic segment in $\widetilde{S}$, where S is regarded as the base surface of a split block B in the tube electrocuted model. Let λ_{B0} denote $\lambda \times \{0\}$. Then for each split component K, $K \cap (S \times i)$ $(i = 0, 1)$ is an *amalgamation component* of $\widetilde{S}$. Thus the induced path metric d_{Gel} on $\widetilde{S} \times i$ $(i = 0, 1)$ is the electric pseudo-metric on $\widetilde{S}$ obtained by electrocuting amalgamation components of $\widetilde{S}$.

Let λ_{Gel} denote the electro-ambient quasigeodesic (cf Lemma 2.7) joining a, b in $(\widetilde{S}, d_{Gel})$. Let λ_{B0} denote $\lambda_{Gel} \times \{0\}$.

The map $\phi : S \times \{0\} \to S \times \{1\}$ taking $(x, 0)$ to $(x, 1)$ is a component preserving diffeomorphism. Let $\tilde{\phi}$ be the lift of ϕ to $\widetilde{S}$ equipped with the electric metric d_{Gel}. Then $\tilde{\phi}$ is an isometry by Lemma 2.9. Let $\widetilde{\Phi}$ denote the induced

map on electro-ambient quasigeodesics, i.e. if $\mu = [x, y] \subset (\widetilde{S}, d_{Gel})$, then $\tilde{\Phi}(\mu) = [\phi(x), \phi(y)]$ is the electro-ambient quasigeodesic joining $\phi(x), \phi(y)$. Let λ_{B1} denote $\Phi(\lambda_{Gel}) \times \{1\}$.

For the universal cover $\widetilde{B}$ of the split block B, define:

$$\mathcal{L}_\lambda(B) = \bigcup_{i=0,1} \lambda_{Bi}$$

Definition: Each $\widetilde{S} \times i$ for $i = 0, 1$ will be called a **horizontal sheet** of $\widetilde{B}$ when B is a split block.

Construction of $\Pi_{\lambda,B}$ - Thick Block

For $i = 0, 1$, let Π_{Bi} denote nearest point projection of $\widetilde{S} \times \{i\}$ onto λ_{Bi} in the path metric on $\widetilde{S} \times \{i\}$.

For the universal cover $\widetilde{B}$ of the thick block B, define:

$$\Pi_{\lambda,B}(x) = \Pi_{Bi}(x), x \in \widetilde{S} \times \{i\}, i = 0, 1$$

Construction of $\Pi_{\lambda,B}$ - Split Block

For $i = 0, 1$, let Π_{Bi} denote nearest point projection of $\widetilde{S} \times \{i\}$ onto λ_{Bi}.

Here the nearest point projection is taken in the sense of the definition preceding Lemma 2.12, i.e. minimizing the ordered pair (d_{Gel}, d_{hyp}) (where d_{Gel}, d_{hyp} refer to electric and hyperbolic metrics respectively.)

For the universal cover $\widetilde{B}$ of the split block B, define:

$$\Pi_{\lambda,B}(x) = \Pi_{Bi}(x), x \in \widetilde{S} \times \{i\}, i = 0, 1$$

$\Pi_{\lambda,B}$ is a coarse Lipschitz retract - Thick Block

The proof for a thick block is exactly as in [Mit98b] and [Mj06a]. We omit it here.

Lemma 5.1. *(Theorem 3.1 of [Mj06a]) There exists $C > 0$ such that the following holds:*
Let $x, y \in \widetilde{S} \times \{0, 1\} \subset \widetilde{B}$ for some thick block B. Then

$$d(\Pi_{\lambda,B}(x), \Pi_{\lambda,B}(y)) \le Cd(x, y).$$

$\Pi_{\lambda,B}$ is a retract - Split Block

Lemma 5.2. *There exists $C > 0$ such that the following holds:*
Let $x, y \in \widetilde{S} \times \{0, 1\} \subset \widetilde{B}$ for some split block B. Then $d_G(\Pi_{\lambda,B}(x), \Pi_{\lambda,B}(y)) \le Cd_G(x, y)$.

Proof: It is enough to show this for the following cases:
1) $x, y \in \widetilde{S} \times \{0\}$ OR $x, y \in \widetilde{S} \times \{1\}$.
This follows directly from Lemma 2.10.
2) $x = (p, 0)$ and $y = (p, 1)$ for some $p \in \widetilde{S}$

First note that $(\widetilde{S}, d_{Gel})$ is uniformly hyperbolic as a metric space (in fact a tree) and $\widetilde{\phi} : \widetilde{S} \times \{0\} \to \widetilde{S} \times \{1\}$ induces an isometry of the d_{Gel} metric by Lemma 2.9 as ϕ is a component preserving diffeomorphism. Case2 now follows from the fact that that quasi-isometries and nearest-point projections almost commute (Lemma 2.11).$\square$

Remark 5.3. Disambiguation: *In the next section, we shall come across the situation where one horizontal surface $\widetilde{S} \times \{i\}$ can occur as the bottom surface of a split block and as the top surface of a thick block, or vice versa. In this case, the nearest point projection could be in either of the two senses:*
a) Projection onto a hyperbolic geodesic $[a, b]$ in the hyperbolic metric on $\widetilde{S}$.
b) Projection onto an electro-ambient quasigeodesic $[a, b]_{ea}$ minimizing the ordered pair (d_{Gel}, d_{hyp}).
Lemma 2.12 now says that the hyperbolic and electric projections of p onto the hyperbolic geodesic $[a, b]$ and the electro-ambient geodesic $[a, b]_{ea}$ respectively 'almost agree': If π_h and π_e denote the hyperbolic and electric projections, then there exists (uniform) $C_1 > 0$ such that $d(\pi_h(p), \pi_e(p)) \leq C_1$.

5.2 Construction of $\mathcal{L}_\lambda$ and Π_λ

Given a manifold M of split geometry, we know that M is homeomorphic to $S \times J$ for $J = [0, \infty)$ or $(-\infty, \infty)$. By definition of split geometry, there exists a sequence I_i of intervals and blocks B_i where the metric on $S \times I_i$ coincides with that on some building block B_i (thick or split). Denote:
- $\mathcal{L}_\mu(B_i) = \mathcal{L}_{i\mu}$
- $\Pi_{\mu, B_i} = \Pi_{i\mu}$

Now for a block $B = S \times I$ (thick or amalgamated), a natural map Φ_B may be defined taking $\mu = \widetilde{B}_{\mu, B} \cap \widetilde{S} \times \{0\}$ to a geodesic $\widetilde{B}_{\mu, B} \cap \widetilde{S} \times \{1\} = \Phi_B(\mu)$. Let the map Φ_{B_i} be denoted as Φ_i for $i \geq 0$. For $i < 0$ we shall modify this by defining Φ_i to be the map that takes $\mu = B_{\mu, B_i} \cap \widetilde{S} \times \{1\}$ to a geodesic $B_{\mu, B_i} \cap \widetilde{S} \times \{0\} = \Phi_i(\mu)$.

We start with a reference block B_0 and a reference geodesic segment $\lambda = \lambda_0$ on the 'lower surface' $\widetilde{S} \times \{0\}$. Now inductively define:
- $\lambda_{i+1} = \Phi_i(\lambda_i)$ for $i \geq 0$
- $\lambda_{i-1} = \Phi_i(\lambda_i)$ for $i \leq 0$
- $\mathcal{L}_{i\lambda} = \mathcal{L}_{\lambda_i}(B_i)$
- $\Pi_{i\lambda} = \Pi_{\lambda_i, B_i}$
- $\mathcal{L}_\lambda = \bigcup_i \mathcal{L}_{i\lambda}$
- $\Pi_\lambda = \bigcup_i \Pi_{i\lambda}$

Recall that each $\widetilde{S} \times i$ for $i = 0, 1$ is called a **horizontal sheet** of $\widetilde{B}$. We will restrict our attention to the union of the horizontal sheets $\widetilde{M_H}$ of $\widetilde{M}$ with the metric induced from the graph model.

Clearly, $\mathcal{L}_\lambda \subseteq \widetilde{M_H} \subset \widetilde{M}$, and Π_λ is defined from $\widetilde{M_H}$ to $\mathcal{L}_\lambda$. Since $\widetilde{M_H}$ is a 'coarse net' in $\widetilde{M}$ (equipped with the *graph metric*), we will be able to get all

the coarse information we need by restricting ourselves to $\widetilde{M_H}$.

By Lemmas 5.1 and 5.2 and by Remark 5.3, we obtain the fact that each $\Pi_{i\lambda}$ is a retract. Hence assembling all these retracts together, we have the following basic theorem:

Theorem 5.4. *There exists $C > 0$ such that for any geodesic $\lambda = \lambda_0 \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, the retraction $\Pi_\lambda : \widetilde{M_H} \to \mathcal{L}_\lambda$ satisfies:*

Then $d_G(\Pi_{\lambda,B}(x), \Pi_{\lambda,B}(y)) \leq C d_G(x,y) + C$.

Note: For Theorem 5.4 above, note that all that we really require is that the universal cover $\widetilde{S}$ be a hyperbolic metric space. There is *no restriction on $\widetilde{M_H}$*. In fact, Theorem 5.4 would hold for general stacks of hyperbolic metric spaces with blocks of split geometry.

5.3 Heights of Blocks

Recall that each thick or split block B_i is identified with $S \times I$ where each fiber $\{x\} \times I$ has length $\leq l_i$ for some l_i, called the *thickness* of the block B_i.

Observation: $\widetilde{M_H}$ is a 'coarse net' in $\widetilde{M}$ in the graph metric, but not in the weld or tube-electrocuted metrics. In the graph model, any point can be connected by a vertical segment of length ≤ 1 to one of the boundary horizontal sheets.

However, there are points within split components which are at a distance of the order of l_i from the boundary horizontal sheets in the hyperbolic metric d or the weld metric d_{wel} or the tube electrocuted metric d_{tel}. Since l_i is arbitrary, $\widetilde{M_H}$ is no longer a 'coarse net' in $(\widetilde{M}, d_{wel})$ or $(\widetilde{M}, d_{tel})$.

Bounded Height of Thick Block

Let $\mu \subset \widetilde{S} \times \{0\} \subset \widetilde{B_i}$ be a geodesic in a (thick or split) block. Then there exists a (K_i, ϵ_i)- quasi-isometry ψ_i ($= \phi_i$ for thick blocks) from $\widetilde{S} \times \{0\}$ to $\widetilde{S} \times \{1\}$ and Ψ_i is the induced map on geodesics. Hence, for any $x \in \mu$, $\psi_i(x)$ lies within some bounded distance C_i of $\Psi_i(\mu)$. But x is connected to $\psi_i(x)$ by

Case 1 - Thick Blocks: a vertical segment of length 1

Case 2 - Split Blocks: the union of

1) a horizontal segment of length bounded by (some uniform) C' (cf. Lemma 2.7) connecting $(x, 0)$ to a point on the electro-ambient geodesic $\mathcal{L}_\lambda(B) \cap \widetilde{S} \times \{0\}$

2) a vertical segment of electric length one in the **graph model** connecting $(x, 0)$ to $(x, 1)$. Such a path has to travel *through a split block* and has length less than l_i, where l_i is the thickness of the ith block B_i.

3) a horizontal segment of length less than C' (Lemma 2.7) connecting $(\phi_i(x), 1)$ to a point on the hyperbolic geodesic $\mathcal{L}_\lambda(B) \cap \widetilde{S} \times \{1\}$

Thus x can be connected to a point $x' \in \Psi_i(\mu)$ by a path of length less than $g(i) = 2C' + l_i$. Recall that λ_i is the geodesic on the lower horizontal surface of the block $\widetilde{B_i}$. The same can be done for blocks $\widetilde{B_{i-1}}$ and *going down* from λ_i to λ_{i-1}. What we have thus shown is:

Lemma 5.5. *There exists a function $g : \mathbb{Z} \to \mathbb{N}$ such that for any block B_i (resp. B_{i-1}), and $x \in \lambda_i$, there exists $x' \in \lambda_{i+1}$ (resp. λ_{i-1}), satisfying:*

$$d(x, x') \leq g(i)$$

6 Recovery

The previous Section was devoted to constructing a quasiconvex ladder in an electric metric. In this section we shall be concerned with recovering information about hyperbolic geodesics from electric ones.

6.1 Admissible Paths

We want to first define a collection of $\mathcal{L}_\lambda$-**elementary admissible paths** lying in a bounded neighborhood of $\mathcal{L}_\lambda$ in the d_G metric. $\mathcal{L}_\lambda$ is not connected. Hence, it does not make much sense to speak of the path-metric on $\mathcal{L}_\lambda$. To remedy this we introduce a 'thickening' (cf. [Gro93]) of $\mathcal{L}_\lambda$ which is path-connected and where the paths are controlled. A $\mathcal{L}_\lambda$-**admissible path** will be a composition of $\mathcal{L}_\lambda$-elementary admissible paths.

First off, admissible paths in the graph model are defined to consist of the following:
1) Horizontal segments along some $\widetilde{S} \times \{i\}$ for $i = \{0, 1\}$.
2) Vertical segments $x \times [0, 1]$, where $x \in \widetilde{S}$.

We shall choose a subclass of these admissible paths to define $\mathcal{L}_\lambda$-elementary admissible paths.

$\mathcal{L}_\lambda$-elementary admissible paths in the thick block
Let $B = S \times [i, i+1]$ be a thick block, where each (x, i) is connected by a vertical segment of length 1 to $(x, i+1)$. Let ϕ be the map that takes (x, i) to $(x, i+1)$. Also Φ is the map on geodesics induced by ϕ. Let $\mathcal{L}_\lambda \cap \widetilde{B} = \lambda_i \cup \lambda_{i+1}$ where λ_i lies on $\widetilde{S} \times \{i\}$ and λ_{i+1} lies on $\widetilde{S} \times \{i+1\}$. Let π_j, for $j = i, i+1$ denote nearest-point projections of $\widetilde{S} \times \{j\}$ onto λ_j. Since ϕ is a quasi-isometry, there exists $C > 0$ such that for all $(x, i) \in \lambda_i$, $(x, i+1)$ lies in a C-neighborhood of $\Phi(\lambda_i) = \lambda_{i+1}$. The same holds for ϕ^{-1} and points in λ_{i+1}, where ϕ^{-1} denotes the *quasi-isometric inverse* of ϕ from $\widetilde{S} \times \{i+1\}$ to $\widetilde{S} \times \{i\}$. The $\mathcal{L}_\lambda$-**elementary admissible paths** in $\widetilde{B}$ consist of the following:
1) Horizontal geodesic subsegments of λ_j, $j = \{i, i+1\}$.
2) Vertical segments of length 1 (both in d_G and d metrics) joining $x \times \{0\}$ to $x \times \{1\}$.
3) Horizontal geodesic segments lying in a C-neighborhood of λ_j, $j = i, i+1$.

$\mathcal{L}_\lambda$-elementary admissible paths in the split block
Let $B = S \times [i, i+1]$ be a split block, where each (x, i) is connected by a segment of d_G length one and hyperbolic length $\leq C(B)$ (due to bounded thickness of B) to $(x, i+1)$. As before we regard ϕ as the map from $\widetilde{S} \times \{i\}$ to $\widetilde{S} \times \{i+1\}$

that is the identity on the first component. Also Φ is the map of electro-ambient quasigeodesics induced by ϕ. Let $\mathcal{L}_\lambda \cap \widetilde{B} = \bigcup_{j=i,i+1} \lambda_j$ where λ_j lies on $\widetilde{S} \times \{j\}$. π_j denotes nearest-point projection of $\widetilde{S} \times \{j\}$ onto λ_j (in the appropriate sense - minimizing the ordered pair of electric and hyperbolic distances). Since ϕ is an electric isometry, but a hyperbolic quasi-isometry, there exists $C > 0$ (uniform constant) and $K = K(B)$ such that for all $x \in \lambda_i$, $\phi(x)$ lies in a (d_G) C-neighborhood and a hyperbolic K-neighborhood of $\Phi(\lambda_i) = \lambda_{i+1}$. The same holds for ϕ^{-1} and points in λ_{i+1}, where ϕ^{-1} denotes the *quasi-isometric inverse* of ϕ from $\widetilde{S} \times \{i+1\}$ to $\widetilde{S} \times \{i\}$.

Again, since λ_i and λ_{i+1} are electro-ambient quasigeodesics, we further note that there exists $C > 0$ (assuming the same *uniform* C for convenience) such that for all $(x,i) \in \lambda_i$, $(x, i+1)$ lies in a (hyperbolic) C-neighborhood of λ_{i+1}.

The $\mathcal{L}_\lambda$-**elementary admissible paths** in $\widetilde{B}$ consist of the following:
1) Horizontal subsegments of λ_j, $j = \{i, i+1\}$.
2) Vertical segments joining $x \times \{i\}$ to $x \times \{i+1\}$. These have hyperbolic 'thickness' $l = l(B)$ and graph thickness one, by Lemma 5.5.
3) Horizontal geodesic segments lying in a *hyperbolic C-neighborhood* of λ_j, $j = i, i+1$.
4) Horizontal hyperbolic segments of *electric length* $\leq C$ and *hyperbolic length* $\leq K(B)$ joining points of the form $(\phi(x), i+1)$ to a point on λ_{i+1} for $x \in \lambda_i$.
5) Horizontal hyperbolic segments of *electric length* $\leq C$ and *hyperbolic length* $\leq K(B)$ joining points of the form $(\phi^{-1}(x), i)$ to a point on λ_i for $x \in \lambda_{i+1}$.

Definition: A $\mathcal{L}_\lambda$-admissible path is a union of $\mathcal{L}_\lambda$-elementary admissible paths.

The next lemma follows from the above definition and Lemma 5.5.

Lemma 6.1. *There exists a function $g : \mathbb{Z} \to \mathbb{N}$ such that for any block B_i, and x lying on a $\mathcal{L}_\lambda$-admissible path in $\widetilde{B_i}$, there exist $y \in \lambda_i$ and $z \in \lambda_{i+1}$ such that*

$$d_{wel}(x, y) \leq g(i)$$
$$d_{wel}(x, z) \leq g(i)$$

Similarly,

$$d(x, y) \leq g(i)$$
$$d(x, z) \leq g(i)$$

where d_{wel} and d are the weld and hyperbolic metrics respectively.

Let $h(i) = \Sigma_{j=0\ldots i} g(j)$ be the sum of the values of $g(j)$ as j ranges from 0 to i (with the assumption that increments are by $+1$ for $i \geq 0$ and by -1 for $i \leq 0$). Then we have from Lemma 6.1 above,

Corollary 6.2. *There exists a function $h : \mathbb{Z} \to \mathbb{N}$ such that for any block B_i, and x lying on a $\mathcal{L}_\lambda$-admissible path in $\widetilde{B_i}$, there exist $y \in \lambda_0 = \lambda$ such that:*

$$d_{wel}(x, y) \leq h(i); \ d(x, y) \leq h(i)$$

Important Note: In the above Lemma 6.1 and Corollary 6.2, it is important to note that the distance d is the **hyperbolic** (not the graph) metric. This is because the lengths occurring in elementary paths of types (4) and (5) above are hyperbolic lengths depending only on i (in B_i).

Next suppose that λ lies outside $B_N(p)$, the N-ball about a fixed reference point p on the boundary horizontal surface $\widetilde{S} \times \{0\} \subset \widetilde{B_0}$. Then by Corollary 6.2, any x lying on a $\mathcal{L}_\lambda$-admissible path in $\widetilde{B_i}$ satisfies

$$d_{wel}(x, p) \geq N - h(i)$$

Also, since the electric, and hence hyperbolic 'thickness' (the shortest distance between its boundary horizontal sheets) is ≥ 1, we get,

$$d_{wel}(x, p) \geq |i|$$

Assume for convenience that $i \geq 0$ (a similar argument works, reversing signs for $i < 0$). Then,

$$d_{wel}(x, p) \geq \min_i \, max\{i, N - h(i)\}$$

Let $h_1(i) = h(i) + i$. Then h_1 is a monotonically increasing function on the integers. If $h_1^{-1}(N)$ denote the largest positive integer n such that $h(n) \leq N$, then clearly, $h_1^{-1}(N) \to \infty$ as $N \to \infty$. We have thus shown:

Lemma 6.3. *There exists a function $M(N) : \mathbb{N} \to \mathbb{N}$ such that $M(N) \to \infty$ as $N \to \infty$ for which the following holds:*
For any geodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B_0}$ and any x on a $\mathcal{L}_\lambda$-admissible path,

$$d_{\widetilde{S}}(\lambda, p) \geq N \Rightarrow d_{wel}(x, p) \geq M(N).$$

where to distinguish between the hyperbolic metrics on $\widetilde{S}$ and $\widetilde{M}$ we use $d_{\widetilde{S}}$ here.

6.2   Joining the Dots

Our strategy in this subsection is:
●1 Start with an (electric) geodesic β_e in $(\widetilde{M}, d_G)$ joining the end-points of $\lambda \subset \widetilde{S} = \widetilde{S} \times \{0\}$.
●2 Replace it by an *admissible quasigeodesic*, i.e. an admissible path that is a quasigeodesic in $(\widetilde{M}, d_G)$.
●3 Project the intersection of the admissible quasigeodesic with the horizontal sheets onto $\mathcal{L}_\lambda$.
●4 The result of step 3 above is disconnected. *Join the dots* using $\mathcal{L}_\lambda$-admissible paths.

The end product is an electric quasigeodesic built up of $\mathcal{L}_\lambda$ admissible paths.

Steps 1 and 2:
• Suppose first that B is thick. Then, since $\widetilde{B}$ has thickness 1, any path lying in a thick block can be perturbed to an admissible path lying in $\widetilde{B}$, changing the length by at most a bounded multiplicative factor.
• For B a split block, we decompose paths into horizontal paths lying in some $\widetilde{S} \times \{j\}$, for $j = 0, 1$ and vertical paths of type (2) (see discussion before Lemma 6.1). This can be done without altering electric length within $\widetilde{S} \times [0,1]$. To see this, project any path $\overline{ab}$ beginning and ending on $\widetilde{S} \times \{0,1\}$ onto $\widetilde{S} \times \{0\}$ along the fibers. To connect this to the starting and ending points a, b, we have to at most adjoin vertical segments through a, b. Note that this does not increase the electric length of $\overline{ab}$, as the electric length is determined by the number of split blocks that $\overline{ab}$ traverses.
• Without loss of generality, we can assume that the electric quasigeodesic is one without back-tracking (as this can be done without increasing the length of the geodesic - see [Far98] for instance).
• Abusing notation slightly, assume therefore that β_e is an admissible electric quasigeodesic without backtracking joining the end-points of λ.
This completes Steps 1 and 2.

Step 3:
• Now act on $\beta_e \cap \widetilde{M_H}$ by Π_λ. From Theorem 5.4, we conclude, by restricting Π_λ to the horizontal sheets of $\widetilde{M_G}$ that the image $\Pi_\lambda(\beta_e)$ is a 'dotted electric quasigeodesic' lying entirely on $\mathcal{L}_\lambda$. This completes step 3.

Step 4:
• Note that since β_e consists of admissible segments, we can arrange so that two nearest points on $\beta_e \cap \widetilde{M_H}$ which are not connected to each other form the end-points of a vertical segment of type (2). Let $\Pi_\lambda(\beta_e) \cap \mathcal{L}_\lambda = \beta_d$, be the dotted quasigeodesic lying on $\mathcal{L}_\lambda$. We want to join the dots in β_d converting it into a **connected** electric quasigeodesic built up of $\mathcal{L}_\lambda$**-admissible paths**.
1) For vertical segments in a thick block joining p, q (say), $\Pi_\lambda(p), \Pi_\lambda(q)$ are a bounded hyperbolic distance apart. Hence, by Lemma 5.1, we can join $\Pi_\lambda(p), \Pi_\lambda(q)$ by a $\mathcal{L}_\lambda$-admissible path of length bounded by some C_0 (independent of B, λ).
2) Vertical segments in a split block $\widetilde{B_i}$ of d_G length one and hyperbolic length $\leq l_i$: Such segments lie within a lift of a split block. The image of such a segment under Π_λ, too, has d_G length one since the projection of any split component lies within a split component.
3) By Remark 5.3 projections of a point in $\widetilde{S} \times \{i\}$ onto a hyperbolic geodesic or an electro-ambient quasigeodesic in $\widetilde{S} \times \{i\}$ joining a pair of points p, q are a uniformly bounded hyperbolic distance apart. Hence, by the proof of Lemma 5.2, we can join them by an $\mathcal{L}_\lambda$-admissible path of length bounded by some uniform C_1 (independent of B_i, λ).

After joining the dots, we can assume further that the quasigeodesic thus obtained does not backtrack.

Putting all this together, we conclude:

Lemma 6.4. *There exists a function $M(N) : \mathbb{N} \to \mathbb{N}$ such that $M(N) \to \infty$ as $N \to \infty$ for which the following holds:*
For any geodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, and a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, there exists a connected electric quasigeodesic β_{adm} without backtracking, such that
* *β_{adm} is built up of $\mathcal{L}_\lambda$-admissible paths.*
* *β_{adm} joins the end-points of λ.*
* *$d(\lambda, p) \geq N \Rightarrow d_{wel}(\beta_{adm}, p) \geq M(N)$.*

Proof: The first two criteria follow from the discussion preceding this lemma. The last follows from Lemma 6.3 since the discussion above gives a quasigeodesic built up out of admissible paths. Note that we make explicit here the fact that the metric on $\widetilde{M}$ is the welded metric. □

6.3 Recovering Electro-ambient Quasigeodesics I

This subsection is devoted to extracting an electro-ambient quasigeodesic β_{ea} in $(\widetilde{M}, d_G)$ from an $\mathcal{L}_\lambda$-admissible quasigeodesic β_{adm}. β_{ea} shall satisfy the property indicated by Lemma 6.4 above.

Lemma 6.5. *There exist κ, ϵ and a function $M'(N) : \mathbb{N} \to \mathbb{N}$ such that $M'(N) \to \infty$ as $N \to \infty$ for which the following holds:*
For any geodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, and a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, there exists a (κ, ϵ) electro-ambient quasigeodesic β_{ea} without backtracking in $(\widetilde{M}, d_G)$, such that
* *β_{ea} joins the end-points of λ.*
* *$d(\lambda, p) \geq N \Rightarrow d_{wel}(\beta_{ea}, p) \geq M'(N)$.*

Proof: From Lemma 6.4, we have an $\mathcal{L}_\lambda$ - admissible quasigeodesic β_{adm} without backtracking and a function $M(N)$ satisfying the conclusions of the Lemma. Since β_{adm} does not backtrack, we can decompose it as a union of non-overlapping segments $\beta_1, \cdots \beta_k$, such that each β_i is either an admissible (hyperbolic) quasigeodesic lying outside split components, or an $\mathcal{L}_\lambda$-admissible quasigeodesic lying entirely within some split component $\widetilde{K_i}$. Further, since β_{adm} does not backtrack, we can assume that all K_i's are distinct.

We modify β_{adm} to an electro-ambient quasigeodesic β_{ea} in $(\widetilde{M}, d_G)$ as follows:
1) β_{ea} coincides with β_{adm} outside split components.
2) There exist κ, ϵ such that if some β_i lies within a split component $\widetilde{K_i}$ then it may be replaced by a (κ, ϵ) ambient quasigeodesic β_i^{ea} (in the intrinsic metric on $\widetilde{K_i}$ joining the end-points of β_i and lying within $\widetilde{K_i}$.

The resultant path β_{ea} is clearly an electro-ambient quasigeodesic without backtracking. Next, each component β_i^{ea} lies in a C_i neighborhood of β_i, where C_i depends only on the thickness l_i of the split component K_i.

We let $C(n)$ denote the maximum of the values of C_i for $K_i \subset B_n$. Then, as in the proof of Lemma 6.3, we have for any $z \in \beta_{ea} \cap B_n$,

$$d(z, p) \geq \max\left(n, M(N) - C(n)\right)$$

Again, as in Lemma 6.3, this gives us a (new) function $M'(N) : \mathbb{N} \to \mathbb{N}$ such that $M'(N) \to \infty$ as $N \to \infty$ for which
- $d(\lambda, p) \geq N \Rightarrow d_{wel}(\beta_{ea}, p) \geq M'(N)$.

This prove the Lemma. $\square$

6.4 Recovering Electro-ambient Quasigeodesics II

This subsection is devoted to extracting an electro-ambient quasigeodesic β_{ea2} in $\widetilde{M_2} = (\widetilde{M}, d_{CH})$ from an electro-ambient quasigeodesic β_{ea} in $\widetilde{M_1} = (\widetilde{M}, d_G)$. β_{ea2} shall satisfy the property indicated by Lemmas 6.4 and 6.5 above. By Remark 4.18 the electro-ambient quasigeodesic β_{ea} constructed in Section 6.3 above for $\widetilde{M_1} = (\widetilde{M}, d_G)$ corresponds to a unique path (which we call β_{ea1}) in $\widetilde{M_2}$ (or $(\widetilde{M}, d)$ obtained by replacing intersections of β_{ea} with tube-electrocuted Margulis tubes by hyperbolic geodesics lying in the corresponding Margulis tubes. From Lemma 4.16, $(\widetilde{M}, d_{CH})$ is quasi-isometric to $(\widetilde{M}, d_G)$. Hence the path β_{ea1} is a quasigeodesic in $\widetilde{M_2}$.

Since β_{ea} lies outside a large $M'(N)$-ball about p in $(\widetilde{M}, d_{wel})$ by Lemma 6.5, it follows that the intersection of β_{ea} with the boundary $\partial\mathbb{T}$ of the lift of any Margulis tube $\mathbb{T}$ lies outside an $M'(N)$-ball about p. Each point $x \in \beta_{ea} \cap \partial\mathbb{T}$ lies on a unique totally geodesic hyperbolic disk $D_x \subset \partial\mathbb{T}$. Also, $\beta_{ea1} \cap \mathbb{T} \subset \bigcup_{x \in \beta_{ea} \cap \partial\mathbb{T}} D_x$ by the convexity of $\bigcup_{x \in \beta_{ea} \cap \partial\mathbb{T}} D_x$. Let the maximum diameter of Margulis tubes intersecting the ith block in $\widetilde{M}$ be t_i. Then $d(\beta_{ea1} \cap \widetilde{B_i}, p) \geq d_{wel}(\beta_{ea} \cap \widetilde{B_i}, p) - t_i \geq M'(N) - t_i$. Now, a reprise of the argument in Lemma 6.3 shows that β_{ea1} lies outside a large about p. We state this explicitly for easy reference.

Lemma 6.6. *There exist κ, ϵ and a function $M'(N) : \mathbb{N} \to \mathbb{N}$ such that $M'(N) \to \infty$ as $N \to \infty$ for which the following holds:*
*For any geodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, and a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, there exists a (κ, ϵ) electro-ambient quasigeodesic β_{ea} without backtracking in $(\widetilde{M}, d_G)$ and a path β_{ea1} **corresponding** to β_{ea} in $(\widetilde{M}, d_{CH})$, such that*
- *β_{ea1} joins the end-points of λ.*
- *$d(\lambda, p) \geq N \Rightarrow d(\beta_{ea1}, p) \geq M'(N)$.*

Recall that $\widetilde{M_2} = (\widetilde{M}, d_{CH})$ denotes $\widetilde{M}$ with the electric metric obtained by electrocuting the convex hulls $CH(\widetilde{K}$ of extended split components $\widetilde{K}$. Also, a

(k, ϵ) electro-ambient quasigeodesic γ in $(\widetilde{M}, d_{CH})$ relative to the collection of $CH(\widetilde{K})$'s is a (k, ϵ) quasigeodesic in $(\widetilde{M}, d_{CH})$ such that in an ordering (from the left) of the convex hulls of (extended) split components that γ meets, each $\gamma \cap CH(\widetilde{K})$ is a (k, ϵ) - quasigeodesic in the hyperbolic metric on $CH(\widetilde{K})$.

To obtain an electro-ambient quasigeodesic β_{ea2} in $(\widetilde{M}, d_{CH})$ from β_{ea1}, first observe that there exists D_0 such that the diameter in the d_G metric $dia_G(\beta_{ea1} \cap CH(\widetilde{K}) \leq D_0$ for any $CH(\widetilde{K})$. This follows from the fact that β_{ea1} is a quasigeodesic in $(\widetilde{M}, d_G)$ and from Lemma 4.16, which says that $(\widetilde{M}, d_{CH})$ and $(\widetilde{M}, d_G)$ are quasi-isometric.

Lemma 6.7. *Let* $\alpha \subset CH(\widetilde{K})$ *be a path of length at most* D_0 *in the* d_G *metric joining* $a, b \in CH(\widetilde{K})$. *Further suppose that* $\alpha \cap \widetilde{C}$ *for any split component* $\widetilde{C}$ *is a geodesic in the intrinsic metric on* $\widetilde{C}$ *and that* $\alpha \cap \mathbb{T}$ *is a hyperbolic geodesic for any lift* $\mathbb{T}$ *of a Margulis tube. Let* $\gamma = [a, b]$ *be the hyperbolic geodesic joining* a, b. *Then there exists* $D_1 = D_1(K)$ *such that* γ *lies in a (hyperbolic)* D_1 *neighborhood of* α.

Proof: By the hypotheses α can be described as the union of at most $2D_0$ pieces $\alpha_1, \cdots, \alpha_j, j \leq 2D_0$ such that each α_i is either a geodesic in the intrinsic metric on $\widetilde{C}$ for some split component $\widetilde{C}$ or a hyperbolic geodesic. If α_i is not already a hyperbolic geodesic, let β_i be the hyperbolic geodesic joining its endpoints. Then $d(\gamma, \cup_i \beta_i) \leq j\delta_0 \leq 2D_0\delta_0$, where δ_0 is the (Gromov) hyperbolicity constant of $\mathbf{H}^3$.

Since α meets a bounded number of split components, there exists C_1 such that each split component $\widetilde{C}$ is C_1-quasiconvex. Note that C_1 depends only on the convex hull $CH(\widetilde{K})$ by graph quasiconvexity (Definition-Theorem 4.19) and the fact that any $CH(\widetilde{K})$ meets the lifts of only a uniformly bounded number of split components. Hence for any $\alpha_i \subset \widetilde{C}$, $d(\alpha_i, \beta_i) \leq C_1$. Choosing $D_1 = C_1 + 2D_0\delta_0$, we are through. $\square$

We are now in a position to obtain the last 'recovery' Lemma of this section. The main part of the argument is again a reprise of the similar argument in Lemma 6.3 (referred to again in Lemma 6.6). We shall recount it briefly for completeness.

Lemma 6.8. *There exist* κ, ϵ *and a function* $M_0(N) : \mathbb{N} \to \mathbb{N}$ *such that* $M_0(N) \to \infty$ *as* $N \to \infty$ *for which the following holds:*
For any geodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, *and a fixed reference point* $p \in \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, *there exists a* (κ, ϵ) *electro-ambient quasigeodesic* β_{ea2} *without backtracking in* $(\widetilde{M}, d_{CH})$, *such that*
• β_{ea2} *joins the end-points of* λ.
• $d(\lambda, p) \geq N \Rightarrow d(\beta_{ea2}, p) \geq M_0(N)$.

Proof: By Lemma 6.6 we have a path α in $(\widetilde{M}, d_{CH})$ corresponding to an electro-ambient quasigeodesic in $(\widetilde{M}, d_G)$ satisfying the conclusions of the Lemma with a function $M'(N) \to \infty$ as $N \to \infty$.

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Let β_{ea2} be an electro-ambient quasigeodesic in $(\widetilde{M}, d_{CH})$ joining the end-points of α obtained by looking at the intersections of α with $CH(\widetilde{K})$ for convex hulls of extended split components $\widetilde{K}$, ordered from the left and replacing maximal such intersections with hyperbolic geodesics in $CH(\widetilde{K})$.

Let $x \in \beta_{ea2} \cap CH(\widetilde{K})$ for an extended split component $\widetilde{K}$. Then by construction of the electro-ambient quasigeodesic β_{ea2} from α and Lemma 6.7 there exists $y \in \alpha \cap CH(\widetilde{K})$ and $D_1 = D_1(K)$ such that $d(x,y) \leq D_1(K)$.

If $x \in \widetilde{B_i}$, then, by uniform graph quasiconvexity (Definition-Theorem 4.19), there exist finitely many extended split components K such that $x \in \cap CH(\widetilde{K})$. let D_i be the maximum value of these $D_1(K)$'s. Hence $x \in \beta_{ea2} \cap \widetilde{B_i} \Rightarrow d(x,p) \geq M'(N) - D_i$. Also, by uniform separatedness of split surfaces, $x \in \widetilde{B_i} \Rightarrow d(x,p) \geq i$. Therefore

$$d(\beta_{ea2}, p) \geq \min_i \max\, (i, M'(N) - D_i)$$

Defining $M_0(N) = \min_i \max\, (i, M'(N) - D_i)$, we are through. $\square$

7 Cannon-Thurston Maps for Surfaces Without Punctures

We note the following properties of the pair $(X, \mathcal{H})$ where X is the graph model of $\widetilde{M}$ and $\mathcal{H}$ consists of the (extended) split components. There exist C, D, Δ such that
1) Each split component is C-quasiconvex by Definition-Theorem 4.19.
2) $\widetilde{M_G} = X_G$ is Δ-hyperbolic Lemma 2.1
3) Given K, ϵ, there exists D_0 such that if γ be a (K, ϵ) hyperbolic quasigeodesic joining a, b and if β be a (K, ϵ) electro-ambient quasigeodesic joining a, b, then γ lies in a D_0 neighborhood of β. This follows from Lemma 2.2.

We shall now assemble the proof of the main Theorem.

Theorem 7.1. *Let M be a simply or doubly degenerate hyperbolic 3 manifold **without parabolics**, homeomorphic to $S \times J$ (for $J = [0, \infty)$ or $(-\infty, \infty)$ respectively). Fix a base surface $S_0 = S \times \{0\}$. Then the inclusion $i : \widetilde{S} \to \widetilde{M}$ extends continuously to a map of the compactifications $\hat{i} : \widehat{S} \to \widehat{M}$. Hence the limit set of $\widetilde{S}$ is locally connected.*

Proof: By Definition-Theorem 4.19, M has split geometry and we may assume that $S_0 \subset B_0$, the first block. Let $(\widetilde{M}, d_{CH})$ and $(\widetilde{M}, d_G)$ be as above. Suppose $\lambda \subset \widetilde{S}$ lies outside a large N-ball about p. By Lemma 6.8 we obtain an electro-ambient quasigeodesic without backtracking β_{ea2} lying outside an $M_0(N)$-ball about p (where $M_0(N) \to \infty$ as $N \to \infty$).

Suppose that β_{ea2} is a (κ, ϵ) electro-ambient quasigeodesic, where the constants κ, ϵ depend on 'the coarse Lipschitz constant' of Π_λ and hence only on $\widetilde{S}$ and $\widetilde{M}$.

From Lemma 2.2 we find that if β^h denote the hyperbolic geodesic in $\widetilde{M}$ joining the end-points of λ, then β^h lies in a (uniform) C' neighborhood of β_{ea2}.

Let $M_1(N) = M_0(N) - C'$. Then $M_1(N) \to \infty$ as $N \to \infty$. Further, the hyperbolic geodesic β^h lies outside an $M_1(N)$-ball around p. Hence, by Lemma 1.2, the inclusion $i : \widetilde{S} \to \widetilde{M}$ extends continuously to a map $\hat{i} : \widehat{S} \to \widehat{M}$.

Since the continuous image of a compact locally connected set is locally connected (see [HY61]) and the (intrinsic) boundary of $\widetilde{S}$ is a circle, we conclude that the limit set of $\widetilde{S}$ is locally connected.

This proves the theorem. □

8 Modifications for Surfaces with Punctures

In this section, we shall describe the modifications necessary to prove Theorem 7.1 for surfaces with punctures.

8.1 Partial Electrocution

Let M be a convex hyperbolic 3-manifold with a neighborhood of the cusps excised. Then the boundary of M is of the form $\sigma \times P$, where P is either an interval or a circle, and σ is a horocycle of some fixed length e_0. In the universal cover $\widetilde{M}$, if we excise (open) horoballs, we are left with a manifold whose boundaries are flat horospheres of the form $\widetilde{\sigma} \times \tilde{P}$. Note that $\tilde{P} = P$ if P is an interval, and $\mathbb{R}$ if P is a circle (the case for a $(Z + Z)$-cusp).

The construction of *partially electrocuted* horospheres below is half way between the spirit of Farb's construction (in Lemmas 2.1, 2.3, where the entire horosphere is coned off), and McMullen's Theorem 2.6 (where nothing is coned off, and properties of *ambient quasigeodesics* are investigated).

In the partially electrocuted case, instead of coning all of a horosphere down to a point we cone only horocyclic leaves of a foliation of the horosphere. Effectively, therefore, we have a cone-line rather a cone-point.

Partial Electrocution of Horospheres
Let Y be a convex simply connected hyperbolic 3-manifold. Let $\mathcal{B}$ denote a collection of horoballs. Let X denote Y minus the interior of the horoballs in $\mathcal{B}$. Let $\mathcal{H}$ denote the collection of boundary horospheres.Then each $H \in \mathcal{H}$ with the induced metric is isometric to a Euclidean product $E^1 \times L$ for an interval $L \subset \mathbb{R}$. Here E^1 denotes Euclidean 1-space. Partially electrocute each H by giving it the product of the zero metric with the Euclidean metric, i.e. on E^1 put the zero metric and on L put the Euclidean metric. The resulting space is essentially what one would get (in the spirit of [Far98]) by gluing to each H the mapping cylinder of the projection of H onto the L-factor.

Much of what follows would go through in the following more general setting (See [MR08] for instance):

1. X is (strongly) hyperbolic relative to a collection of subsets H_α, thought of as horospheres (and *not horoballs*).

2. For each H_α there is a uniform large-scale retraction $g_\alpha : H_\alpha \to L_\alpha$ to some (uniformly) δ-hyperbolic metric space L_α, i.e. there exist $\delta, K, \epsilon > 0$ such that for all H_α there exists a δ-hyperbolic L_α and a map $g_\alpha : H_\alpha \to L_\alpha$ with $d_{L_\alpha}(g_\alpha(x), g_\alpha(y)) \le K d_{H_\alpha}(x, y) + \epsilon$ for all $x, y \in H_\alpha$.

3. The coned off space corresponding to H_α is the (metric) mapping cylinder for the map $g_\alpha : H_\alpha \to L_\alpha$.

In Farb's construction L_α is just a single point. The metric, and geodesics and quasigeodesics in the partially electrocuted space will be referred to as the partially electrocuted metric d_{pel}, and partially electrocuted geodesics and quasigeodesics respectively. In this situation, we conclude as in Lemma 2.1:

Lemma 8.1. (X, d_{pel}) *is a hyperbolic metric space and the sets L_α are uniformly quasiconvex.*

Note 1: When K_α is a point, the last statement is a triviality.
Note 2: (X, d_{pel}) is strongly hyperbolic relative to the sets $\{L_\alpha\}$. In fact the space obtained by electrocuting the sets L_α in (X, d_{pel}) is just the space (X, d_e) obtained by electrocuting the sets $\{H_\alpha\}$ in X.
Note 3: The proof of Lemma 8.1 and other such results below follow Farb's [Far98] constructions (See [MP07] for details). For instance, consider a hyperbolic geodesic η in a convex complete simply connected hyperbolic 3-manifold X. Let H_i, $i = 1 \cdots k$ be the partially electrocuted horoballs it meets. Let $N(\eta)$ denote the union of η and H_i's. Let Y denote X minus the interiors of the H_i's. The first step is to show that $N(\eta) \cap Y$ is quasiconvex in (Y, d_{pel}). To do this one takes a hyperbolic R-neighborhood of $N(\eta)$ and projects (Y, d_{pel}) onto it, using the hyperbolic projection. It was shown by Farb in [Far98] that the projections of all horoballs are uniformly bounded in hyperbolic diameter. (This is essentially mutual coboundedness). Hence, given K, choosing R large enough, any path that goes out of an R-neighborhood of $N(\eta)$ cannot be a K-partially electrocuted quasigeodesic. This is the one crucial step that allows the results of [Far98], in particular, Lemma 8.1 to go through in the context of partially electrocuted spaces.

As in Lemma 2.3, partially electrocuted quasigeodesics and geodesics without backtracking have the same intersection patterns with *horospheres and boundaries of lifts of tubes* as electric geodesics without backtracking. Further, since electric geodesics and hyperbolic quasigeodesics have similar intersection patterns with *horoballs and lifts of tubes* it follows that partially electrocuted quasigeodesics and hyperbolic quasigeodesics have similar intersection patterns with *horospheres and boundaries of lifts of tubes*. We state this formally below:

Lemma 8.2. *Given $K, \epsilon \geq 0$, there exists $C > 0$ such that the following holds: Let γ_{pel} and γ denote respectively a (K, ϵ) partially electrocuted quasigeodesic in (X, d_{pel}) and a hyperbolic (K, ϵ)-quasigeodesic in (Y, d) joining a, b. Then $\gamma \cap X$ lies in a (hyperbolic) C-neighborhood of (any representative of) γ_{pel}. Further, outside of a C-neighborhood of the horoballs that γ meets, γ and γ_{pel} track each other.*

Next, we note that partial electrocution preserves quasiconvexity. Suppose that $A \subset Y$ as also $A \cap H$ for all $H \in \mathcal{H}$ are C-quasiconvex. Then given $a, b \in A \cap X$, the hyperbolic geodesic λ in X joining a, b lies in a C-neighborhood of A. Since horoballs are convex, λ cannot backtrack. Let λ_{pel} be the partially electrocuted geodesic joining $a, b \in (X, d_{pel})$. Then by Lemma 8.2 above, we conclude that for all $H \in \mathcal{H}$ that λ intersects, there exist points of λ_{pel} (hyperbolically) near the entry and exit points of λ with respect to H. Since these points lie near $A \cap H$, and since the corresponding L is quasiconvex in (X, d_{pel}), we conclude that λ_{pel} lies within a bounded distance from A near horoballs. For the rest of λ_{pel} the conclusion follows from Lemma 8.2. We conclude:

Lemma 8.3. *Given C_0 there exists C_1 such that if $A \subset Y$ and $A \cap H$ are C_0-quasiconvex for all $H \in \mathcal{H}$, then (A, d_{pel}) is C_1-quasiconvex in (X, d_{pel}).*

8.2 Split geometry for Surfaces with Punctures

Step 1: For a hyperbolic surface S^h (possibly) with punctures, we fix a (small) e_0, and excise the cusps leaving horocyclic boundary components of (ordinary or Euclidean) length e_0. We then take the induced *path metric* on S^h minus cusps and call the resulting surface S. This induced path metric will still be referred to as the hyperbolic metric on S (with the understanding that now S possibly has boundary).

Step 2: The definitions and constructions of **split building blocks** and **split components** now go through with appropriate changes. The only difference is that S now might have boundary curves of length e_0. For thick blocks, we assume that a thick block is the universal curve over a Teichmuller geodesic (of length less than D for some uniform D) minus *cusps* $\times I$.

There is one subtle point about global quasiconvexity (in $\widetilde{M}$) of split components. This does not hold in the metric obtained by merely excising the cusps and equipping the resulting horospheres with the Euclidean metric. What we demand is that each split component *union* the parts of the horoballs that meet its boundary (horocycle times closed interval)'s be quasiconvex in $\tilde{M}$. When we partially electrocute horospheres below, and consider quasiconvexity in the resulting partially electrocuted space, split components in this sense remain quasiconvex by Lemma 8.3.

Step 3: Next, we modify the metric on B by partially electrocuting its boundary components so that the metric on the boundary components of each block $S \times I$ is the product of the zero metric on the horocycles of fixed (Euclidean) length e_0 and the Euclidean metric on the I-factor. The resulting blocks will be called **partially electrocuted blocks**. We require that in the model M_{pel}

obtained by gluing together partially electrocuted blocks, the split components are uniformly quasiconvex. By Lemma 8.3, this follows from quasiconvexity of split components in the sense of the discussion in Step 2 above. Note that M_{pel} may also be constructed directly from M by excising a neighborhood of the cusps and partially electrocuting the resulting horospheres. By Lemma 8.1 $\widetilde{M}_{pel}$ is a hyperbolic metric space.

Step 4: Again, the definitions and constructions of **split blocks** and **split components** go through *mutatis mutandis* for the partially electrocuted manifold $\widetilde{M}_{pel}$. *By Lemma 8.3 quasiconvexity of split components as well as lifts of Margulis tubes is preserved by partial electrocution.*

Step 5: Next, let λ^h be a hyperbolic geodesic in $\tilde{S}^h$. We replace pieces of λ^h that lie within horodisks by shortest horocyclic segments joining its entry and exit points (into the corresponding horodisk). Such a path is called a horo-ambient quasigeodesic in [Mj09]. See Figure below:
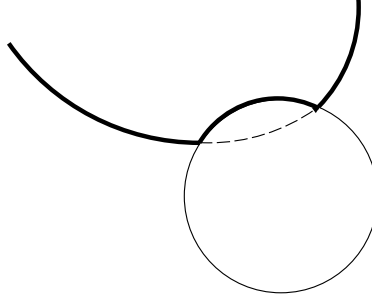

Figure 5:*Horo-ambient quasigeodesic*

Not much changes if we electrocute horocycles and consider electro-ambient quasigeodesics. Geodesics and quasigeodesics without backtracking then travel for free along the zero metric horocycles. This does not change matters much as the geodesics and quasigeodesics in the two resulting constructions track each other by Lemma 2.3.

Step 6: Thus, our starting point for the construction of the hyperbolic ladder $\mathcal{L}_\lambda$ is not a hyperbolic geodesic λ^h but a horoambient quasigeodesic λ.

Step 7: The construction of $\mathcal{L}_\lambda, \Pi_\lambda$ and their properties go through *mutatis mutandis* and we conclude that $\mathcal{L}_\lambda$ is quasiconvex in the *graph metric* $(\widetilde{M}_{pel}, d_G)$ of the partially electrocuted space $\widetilde{M}_{pel}$. As before, $\widetilde{M}_{Hpel}$ will denote the collection of horizontal sheets. The modification of Theorem 5.4 is given below:

Theorem 8.4. *There exists $C > 0$ such that for any horo-ambient geodesic $\lambda = \lambda_0 \subset \tilde{S} \times \{0\} \subset \widetilde{B_0}$, the retraction $\Pi_\lambda : \widetilde{M}_{Hpel} \to \mathcal{L}_\lambda$ satisfies:*

$$d_G(\Pi_{\lambda,B}(x), \Pi_{\lambda,B}(y)) \leq Cd_G(x,y) + C.$$

Step 8: From this step on, the modifications for punctured surfaces follow [Mj09] As in [Mj09], we decompose λ into portions λ^c and λ^b that lie along

horocycles and those that do not. Accordingly, we decompose $\mathcal{L}_\lambda$ into two parts $\mathcal{L}_\lambda^c$ and $\mathcal{L}_\lambda^b$ consisting of parts that lie along horocycles and those that do not. Dotted geodesics and admissible paths are constructed as before. As in Lemma 6.3, we get

Lemma 8.5. *There exists a function $M(N) : \mathbb{N} \to \mathbb{N}$ such that $M(N) \to \infty$ as $N \to \infty$ for which the following holds:*
For any horo-ambient quasigeodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B}_0$, a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B}_0$ and any x on $\mathcal{L}_\lambda^b$,

$$d(\lambda^b, p) \geq N \Rightarrow d(x, p) \geq M(N).$$

Step 9: Construct a 'dotted' ambient electric quasigeodesic lying on $\mathcal{L}_\lambda$ by projecting some(any) ambient electric quasigeodesic onto $\mathcal{L}_\lambda$ by Π_λ. Join the dots using admissible paths to get a connected ambient electric quasigeodesic β_{amb}.

Step 10 Construct from $\beta_{amb} \subset \widetilde{M}$ an electric quasigeodesic γ in $\widetilde{M}_{pel}$ as in Sections 6.3 and 6.4 and note that parts of γ not lying along horocycles lie close to $\mathcal{L}_\lambda^b$.

Step 11 Conclude that if λ^h lies outside large balls in S^h then each point of γ lying outside partially electrocuted horospheres also lies outside large balls.

Step 12 Let γ^h denote the hyperbolic geodesic in $\widetilde{M}^h$ joining the end-points of γ. By Lemma 8.2 γ and γ^h track each other off a bounded (hyperbolic) neighborhood of the electrocuted horoballs. Let X denote $\widetilde{M}^h$ minus interiors of horoballs. Then, every point of $\gamma^h \cap X$ must lie close to some point of γ lying outside partially electrocuted horospheres. Hence from Step (11), if λ^h lies outside large balls about p in S^h then $\gamma^h \cap X$ also lies outside large balls about p in X. In particular, γ^h enters and leaves horoballs at large distances from p. From this it follows (See Theorem 5.9 of [Mj09] for instance) that γ^h lies outside large balls. Hence by Lemma 1.2 there exists a Cannon-Thurston map and the limit set is locally connected.

We state the conclusion below:

Theorem 8.6. *Let M^h be a simply or doubly degenerate 3 manifold homeomorphic to $S^h \times J$ (for $J = [0, \infty)$ or $(-\infty, \infty)$ respectively) for S^h a finite volume hyperbolic surface such that $i : S^h \to M^h$ is a proper map inducing a homotopy equivalence. Then the inclusion $i : \widetilde{S}^h \to \widetilde{M}^h$ extends continuously to a map $\hat{i} : \widehat{S}^h \to \widehat{M}^h$. Hence the limit set of $\widetilde{S}^h$ is locally connected.*

8.3 Local Connectivity of Connected Limit Sets

Here we shall use a Theorem of Anderson and Maskit [AM96] along with Theorems 7.1 and 8.6 above to prove that connected limit sets are locally connected. The connection between Theorems 7.1 and 8.6 and Theorem 8.8 below via Theorem 8.7 is similar to that discussed by Bowditch in [Bow07].

Theorem 8.7. Anderson-Maskit [AM96] *Let* Γ *be a finitely generated Kleinian group with connected limit set. Then the limit set* $\Lambda(\Gamma)$ *is locally connected if and only if every simply degenerate surface subgroup of* Γ *without accidental parabolics has locally connected limit set.*

Combining Theorems 7.1 and 8.6 with Theorem 8.7, we have the following.

Theorem 8.8. *Let* Γ *be a finitely generated Kleinian group with connected limit set* Λ. *Then* Λ *is locally connected.*

In [Mj07], we prove further that the point pre-images of the Cannon-Thurston map for closed surface groups without accidental parabolics are precisely the end-points of leaves of the ending lamination. Extending this last statement to surfaces with punctures is the subject of work in progress.

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